Local Backbones

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Backbones in propositional theories

A backbone of a propositional theory is a variable that has the same truth value in each satisfying assignment.

- i.e., $x \in \mathrm{Var}(\varphi)$ is a backbone of a CNF formula φ if $\varphi \models x$ or $\varphi \models \neg x$.

Identifying backbones allows us to simplify the theory.

Unfortunately, deciding whether a variable is a backbone is coNP-complete.

Our approach:

- Relax and localize the notion of a backbone.
- It is reasonable that some variables are enforced locally (local backbones).
- Main theoretical tool: parameterized complexity theory.

Overview

What are local backbones?

Do local backbones occur?

Parameterized complexity results

Iterative local backbones

What are local backbones?

Definition (k-backbones).

A *k-backbone* of a CNF formula φ is a variable $x \in \mathrm{Var}(\varphi)$ such that for some $\varphi' \subseteq \varphi$ with $|\varphi'| \leq k$ it holds that $\varphi' \models x$ or $\varphi' \models \neg x$.

Example: x_2 is a 2-backbone of φ
($\neg x_2$ is implied by a subset of size 2).

$$\varphi = \{\{x_1, \neg x_2\}, \{\neg x_1, \neg x_2\}, \{x_2, x_3, x_4\}, \{x_2, \neg x_3, x_4\}, \{\neg x_4, x_5\}\}$$

- Every k-backbone of φ is a backbone of φ.
- 1-backbones correspond to unit clauses.

Where were we?

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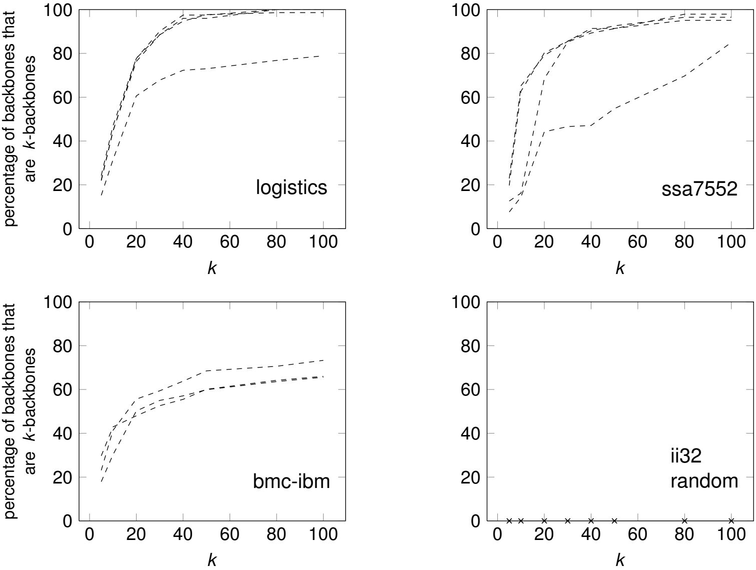
Distribution of local backbones



Parameterized complexity theory

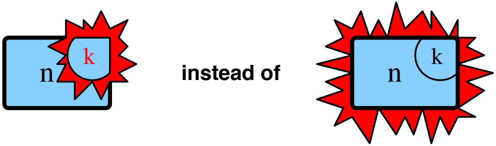
Parameterized complexity theory investigates how to algorithmically exploit structure in problem instances.

- Takes into account a parameter k of the input, besides the input size n.

If k is a constant, then finding k-backbones can be done in polynomial time.

- Brute force search in roughly n^k time (XP).
- For $k = 3, 4, \ldots$ this is already not so practical.

We would like to solve the problem in $f(k) \cdot n^c$ time, for some function f and some constant c: fixed-parameter tractability (FPT).



Parameterized complexity theory

To give evidence that some problems are not FPT,
there exist fixed-parameter intractability classes:

$$\text{FPT} \subseteq \text{W[1]} \subseteq \text{W[2]} \subseteq \cdots \subseteq \text{W[P]}$$

The classes W[t] are based on the question whether certain Boolean circuits are satisfiable with k input nodes set to true.

These classes are not fixed-parameter tractable unless the Exponential Time Hypothesis (ETH) fails.

- ETH: 3SAT cannot be solved in subexponential time.

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The parameterized decision problem

We consider the following parameterized decision problems, for propositional languages $\mathcal{C}$.

LOCAL-BACKBONE[$\mathcal{C}$]

Instance:	*a CNF formula $\varphi \in \mathcal{C}$, a variable $x \in Var(\varphi)$, and an integer $k \geq 1$.*
Parameter:	*k.*
Question:	*Is x a k-backbone of φ?*

Formulas with bounded variable occurrence

Fix an integer $d \geq 1$. We let VO_d denote the class of CNF formulas in which each variable occurs at most d times.

Theorem. LOCAL-BACKBONE(VO_d) is FPT.

Proof (idea). Bounded search tree.

Search for a subset $\varphi' \subseteq \varphi$ witnessing $\varphi' \models \ell$ for some $\ell \in \{x, \neg x\}$ with a bounded search tree.

Start with some clause c containing x.

For each variable y in the current set φ', guess a (non-empty) subset of clauses containing y.

- bounded number of branches, since y occurs in at most d clauses

The depth of the search tree is at most k, since $|\varphi'| \leq k$.

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$$\{\{x_2, x_3\}\}$$

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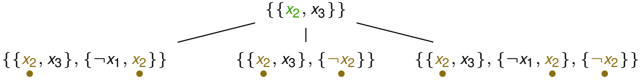

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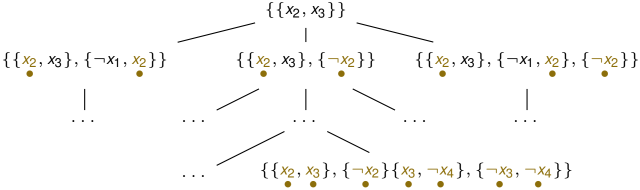

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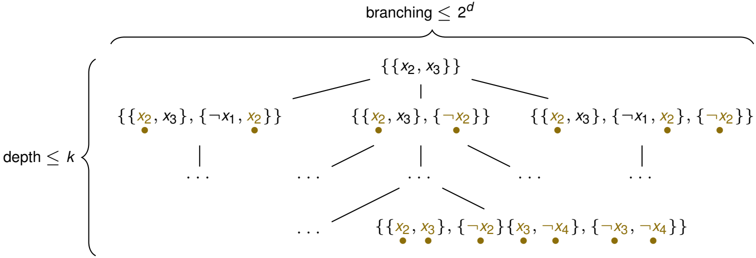
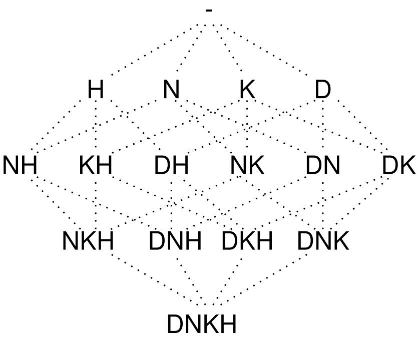


Local backbones of various propositional fragments

Complexity of LOCAL-BACKBONE[$\mathcal{C}$], for $\mathcal{C} \subseteq \{D,N,K,H\}$:



D: no purely negative clauses

N: no unit clauses

K: clauses are Krom

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e.g., DH corresponds to definite Horn

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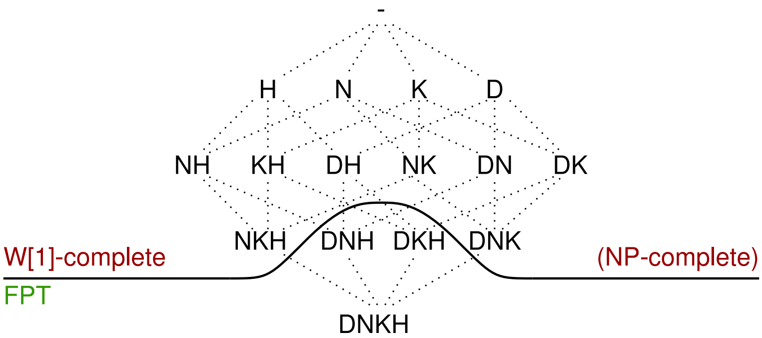

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(All results hold also for the restriction to 3CNF.)

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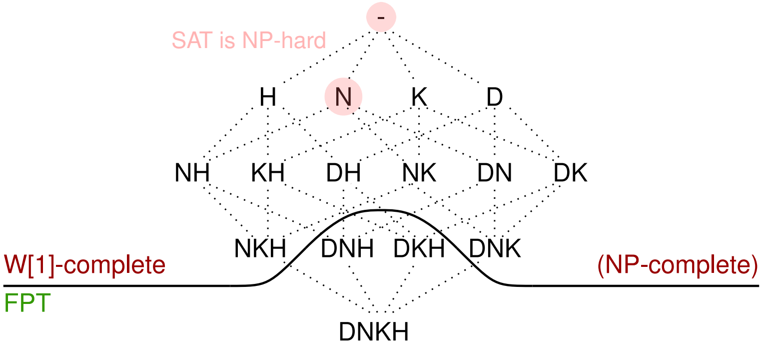

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Small Unsatisfiable Subsets

Local backbones are closely related to small unsatisfiable subsets.

- useful for the repair of inconsistent knowledge bases.

Originally considered in Fellows et al. (2006).

SMALL-UNSATISFIABLE-SUBSET[$\mathcal{C}$]

Instance:	*a CNF formula $\varphi \in \mathcal{C}$, and an integer $k \geq 1$.*
Parameter:	*k.*
Question:	*Is there an unsatisfiable $\varphi' \subseteq \varphi$ with at most k clauses?*

Theorem.

For any $\mathcal{C}$, SMALL-UNSATISFIABLE-SUBSET[$\mathcal{C}$] has the same parameterized complexity as LOCAL-BACKBONE[$\mathcal{C}$].

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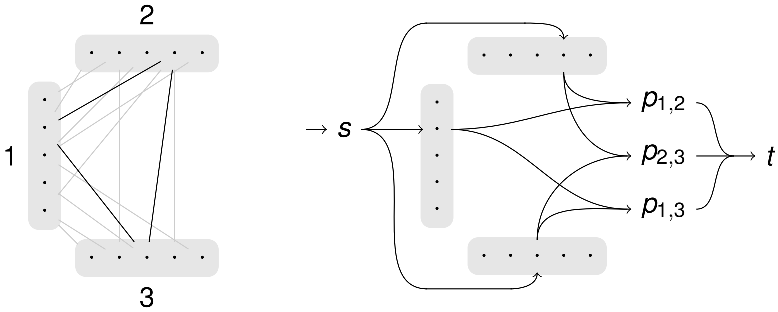
Definite Horn Formulas

Theorem. LOCAL-BACKBONE[DefHorn] is W[1]-hard.

Proof (idea). Reduction from MULTICOLORED-CLIQUE (see below).



(A slight modification of the proof works for the case of NH.)

Krom Formulas

Remember: deciding whether a variable is a (non-local) backbone of a Krom formula φ can be done in poly-time.

- e.g., by using reachability in the implication graph of φ.

Theorem. LOCAL-BACKBONE[Krom] is W[1]-hard.

Proof (idea). Reduction from CLIQUE.

Essential to the proof:

- paths in the implication graph may use some clauses twice,
- so k-reachability in the implication graph cannot be used.

This contrasts to the result of Buresh-Oppenheim & Mitchell (2006,2007) that finding a minimum (tree-like) resolution refutation of a Krom formula can be found in poly-time.

- The smallest refutation does not necessarily use the smallest number of clauses.

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Iterative local backbones

Iterative Local Backbones

Iterative local backbones are variables with an enforced truth value that can be found after iteratively instantiating local backbones.

Definition. Iterative k-backbones.

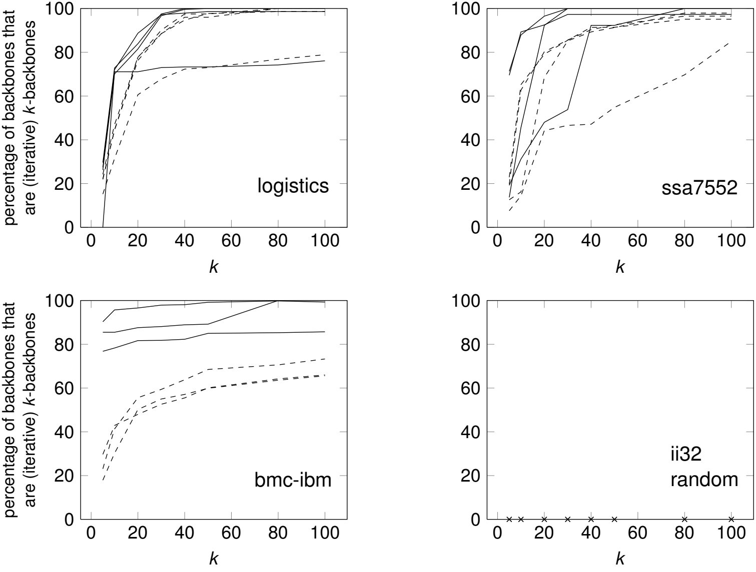
An *iterative k-backbone* of a CNF formula φ is a variable $x \in \mathrm{Var}(\varphi)$ such that either:

- x is a k-backbone of φ; or
- there exists a k-backbone y of φ, with enforced literal $\ell \in \{y, \neg y\}$, and x is an iterative k-backbone of $\varphi|_\ell$.

Example: x_4 is an iterative 2-backbone of φ
($\neg x_2$ is implied by a subset of φ of size 2;
x_4 is implied by a subset of $\varphi|_{\neg x_2}$ of size 2).

$$\varphi = \{\{x_1, \neg x_2\}, \{\neg x_1, \neg x_2\}, \{\cancel{x_2}, x_3, x_4\}, \{\cancel{x_2}, \neg x_3, x_4\}, \{\neg x_4, x_5\}\}$$

Distribution of (iterative) local backbones



Dashed: local backbones, solid: iterative local backbones.

Iterative Local Backbones

For propositional languages $\mathcal{C}$, the parameterized decision problems ITERATIVE-LOCAL-BACKBONE[$\mathcal{C}$] are defined analogously to LOCAL-BACKBONE[$\mathcal{C}$].

Theorem. For any $\mathcal{C}$, if LOCAL-BACKBONE[$\mathcal{C}$] is FPT, then also ITERATIVE-LOCAL-BACKBONE[$\mathcal{C}$] is FPT.

Proof (idea). Iteratively find k-backbones and instantiate them, until a fixed-point is reached.

Theorem. ITERATIVE-LOCAL-BACKBONE[NH] is W[1]-hard.

Proof (idea). The hardness proof for LOCAL-BACKBONE[NH] also works for this case.

Iterative Local Backbones

ITERATIVE-LOCAL-BACKBONE[Krom] is in P.

Proof (idea). Iterative *k*-backbones of a Krom formula φ can be found by iteratively applying backbones that are based on *k*-reachability in the implication graph of φ.

ITERATIVE-LOCAL-BACKBONE[DefHorn] is in P.

Proof (idea). The set of iterative *k*-backbones of a definite Horn formula φ coincides with the set of (non-local) backbones of φ.

!! Remember, LOCAL-BACKBONE[Krom] and LOCAL-BACKBONE[DefHorn] are W[1]-hard

Take home message – quick overview

- Relatively many backbones might be local backbones (or iterative local backbones).
- Identifying local backbones is in XP (poly-time for fixed k).
- For formulas with bounded variable occurrences, it is fixed-parameter tractable.
- It is W[1]-hard already for definite Horn and Krom formulas;
  - interestingly, in these cases iterative local backbones are easier to find (poly-time).
- Finding small unsatisfiable subsets is of the same parameterized complexity (for all fragments).