Exploiting Partial Duality in QBF

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Outline

- Background
  - Dual propagation
  - Dual propagation in existing CNF solvers
- Partial duality
  - Using reconstructed gates
  - Reconstructing Plaisted-Greenbaum
- Experiments and conclusion

CNF representation

- Handles conflicts well, but loses information about solutions

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- Example:

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$$\vee f\left(x_{1,} x_{2,} x_{3,} x_{4,} e\right)$$

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Tseitin

$$\exists e \forall x_1 x_2 x_3 \ldots x_n \ \exists g_1 g_2 g_3 \ldots$$

$$\left(g_1 \vee g_2 \vee g_3 \vee \ldots\right)$$

$$g_1 \equiv \left(x_1 \neq x_2\right)$$

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$$g_k \equiv f\left(x_{1,} \ldots, e\right)$$

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Possible solution:

$$\left(x_1 \wedge \neg x_2\right)$$

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Not verifiable in Tseitin

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- In particular, it would have to include all the x_i

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- Don't care propagation

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May get a solution

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BUT NOT IN GENERAL

Example:

$$\exists e \forall x_1 x_2 x_3 \ldots x_n$$
$$(x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus \ldots \oplus x_n) \vee f(x_{1,} x_{2,} x_{3,} x_{4,} e)$$

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- Such is not a problem for conflicts, because we have Tseitin variables
- But they are not useful in cubes
- So: we lose the ability to generalize solutions

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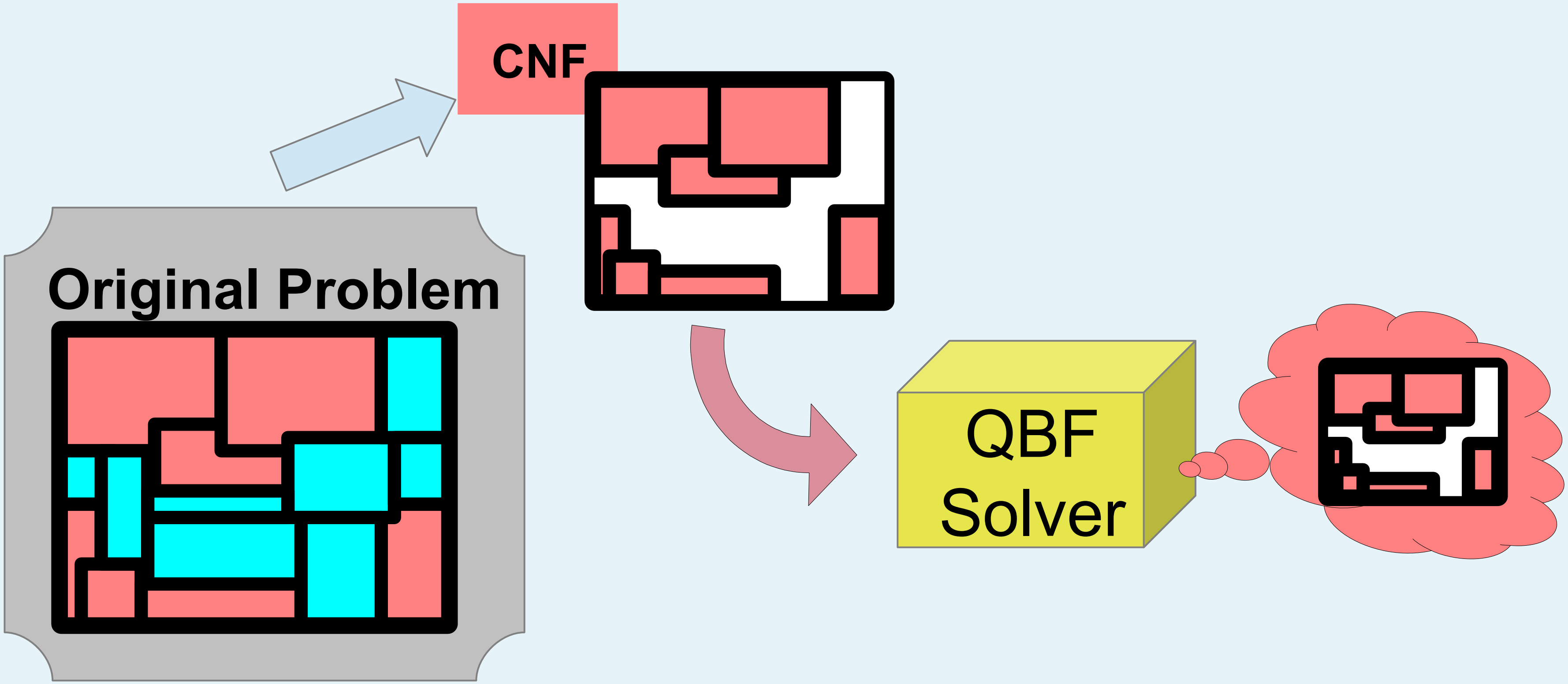
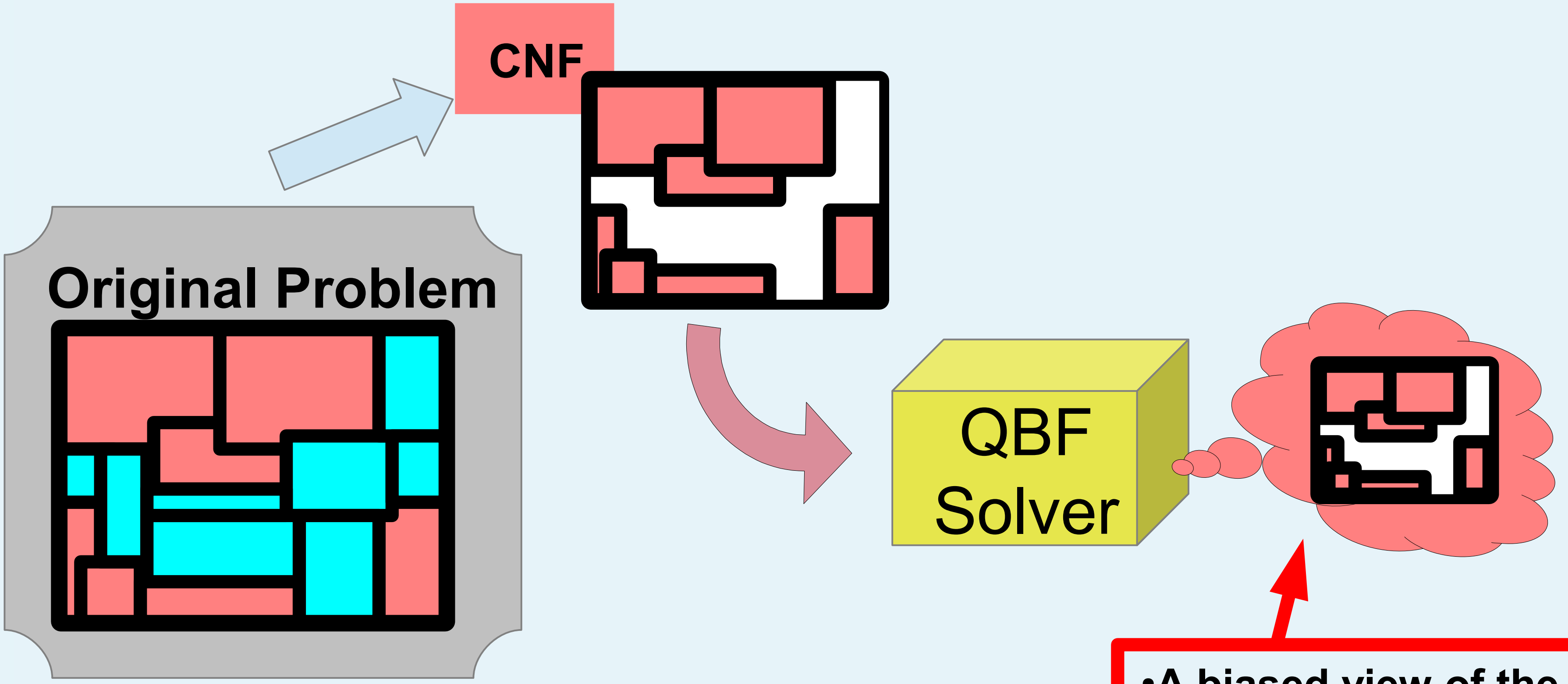
QBF Solving



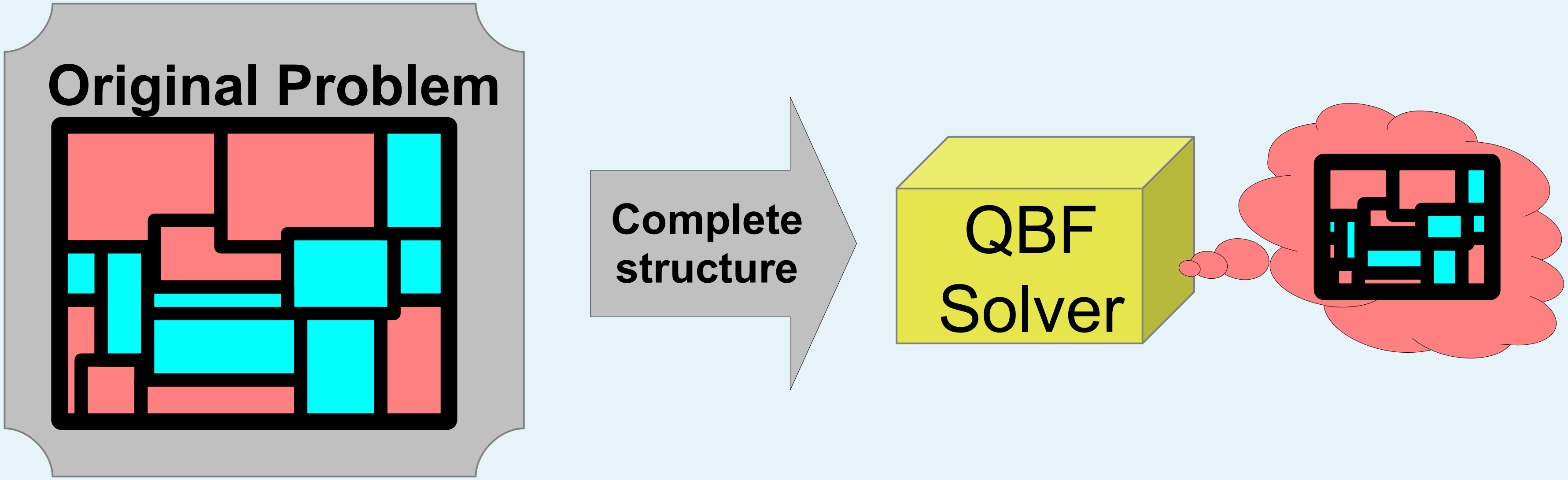
QBF Solving



- A biased view of the problem

QBF Solving

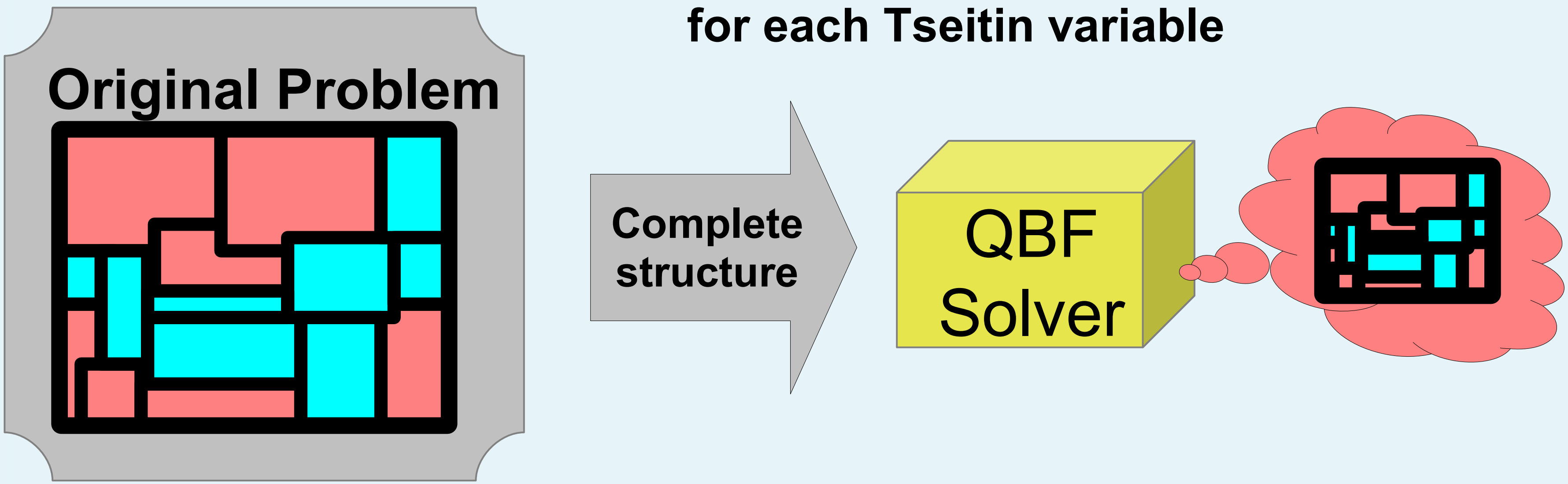
One solution: non-CNF



QBF Solving

One solution: non-CNF

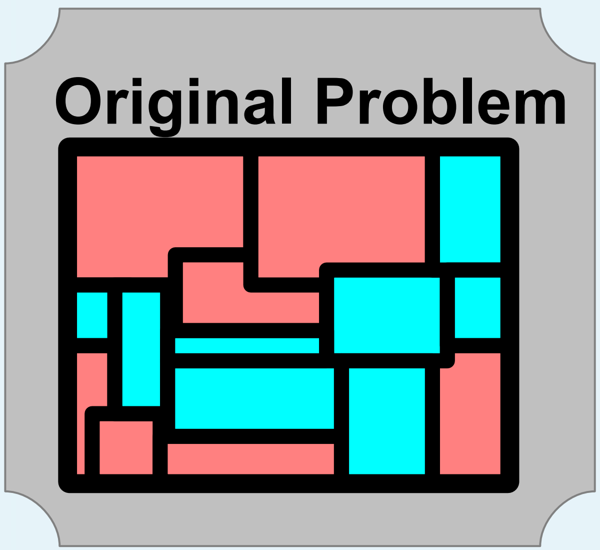
- Implicitly creates a universal copy for each Tseitin variable

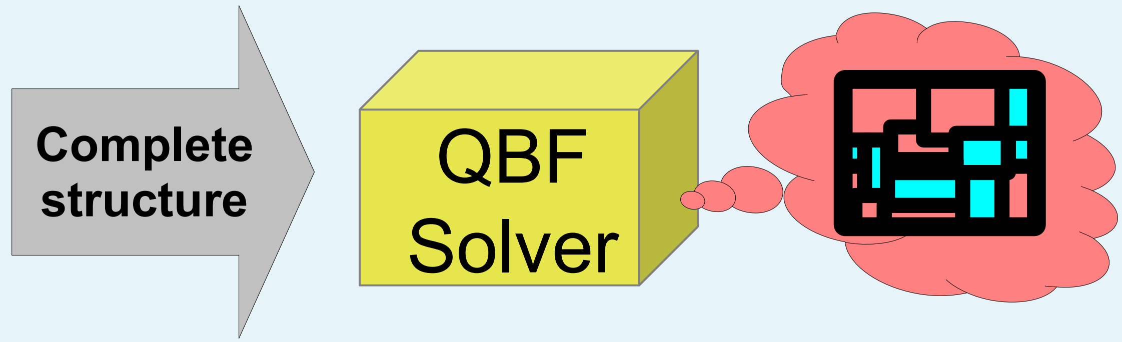


QBF Solving

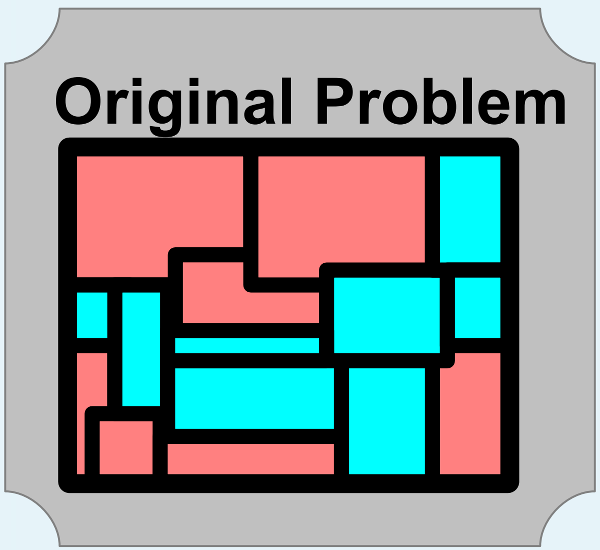
One solution: non-CNF

- Loses implementation efficiency
- Needs specialized techniques
- Limited benefits from other work

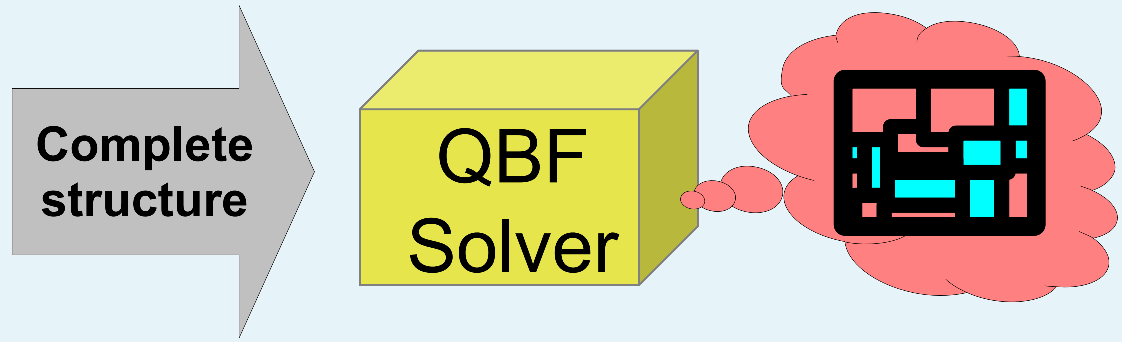




QBF Solving

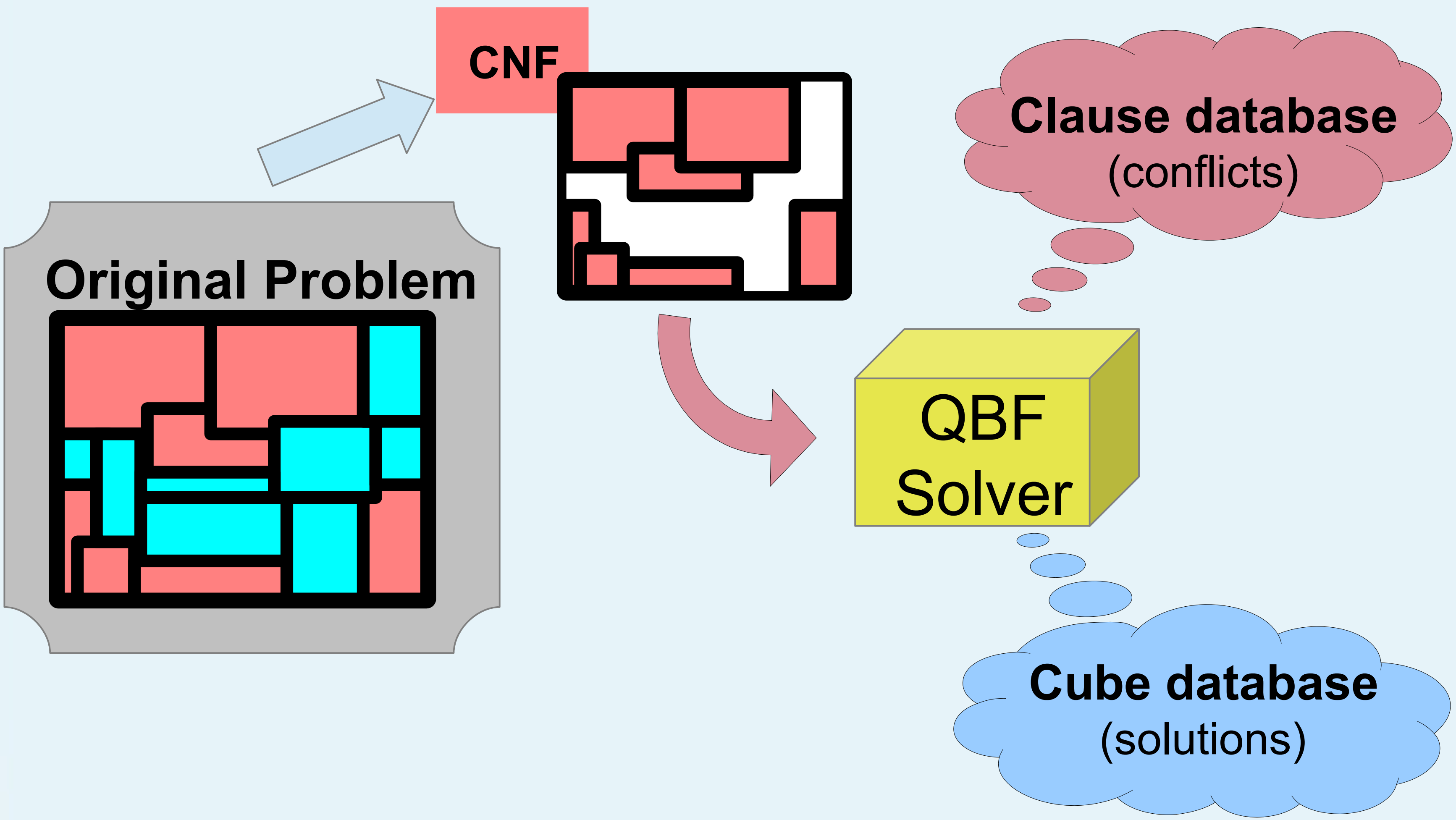


One solution: non-CNF
- **Loses implementation efficiency**
- **Needs specialized techniques**
- **Limited benefits from other work**

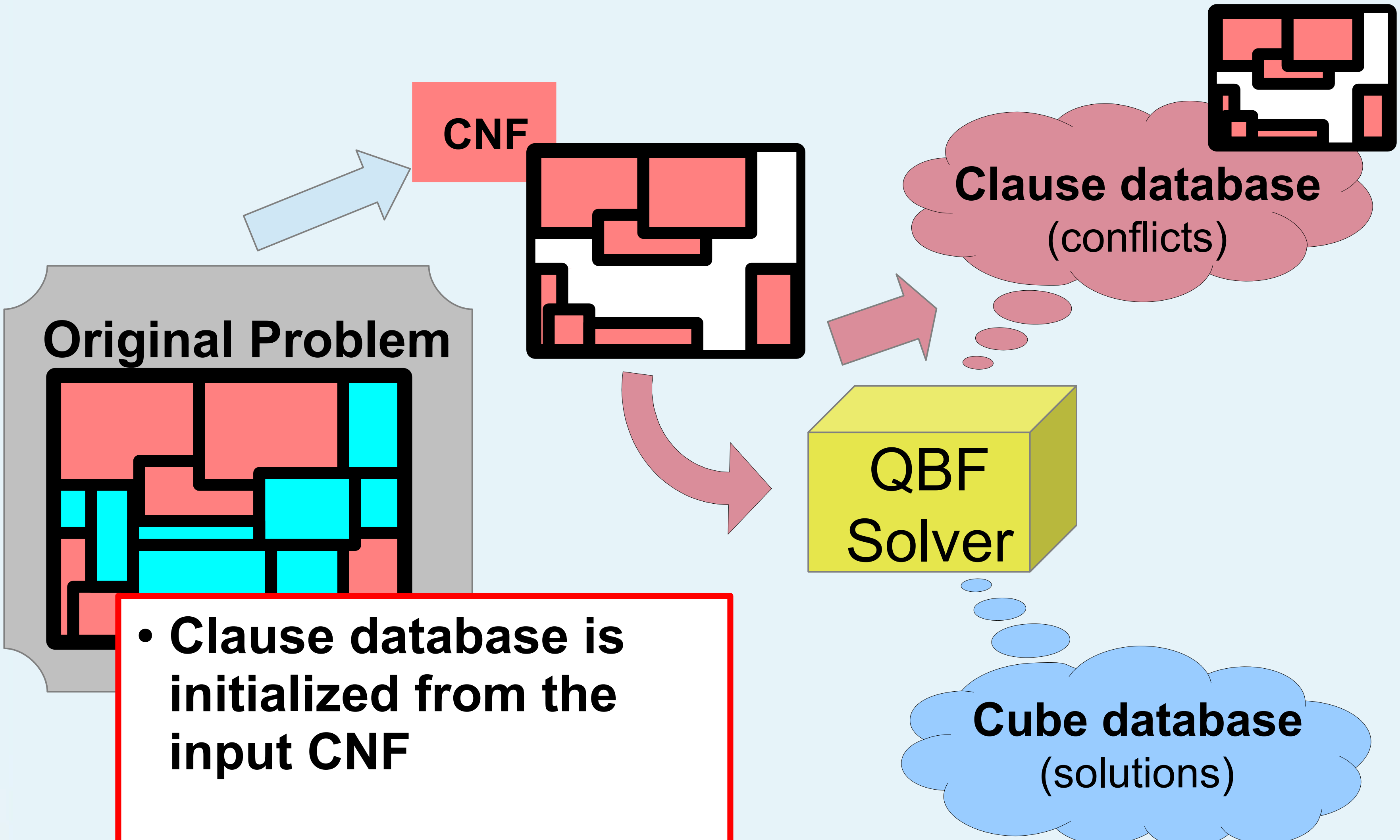


Specialized solvers are unnecessary.

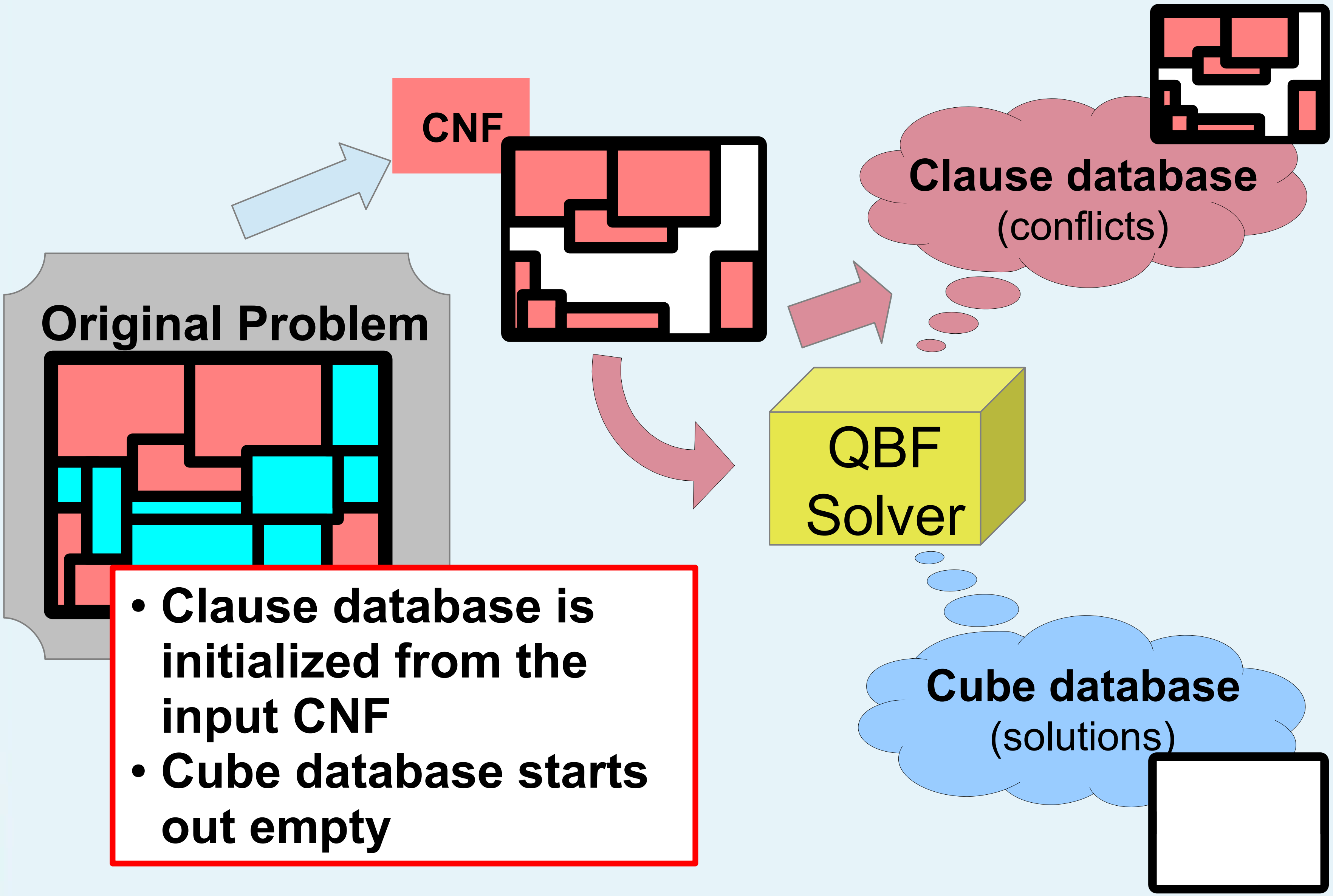
Existing search-based solvers have all the needed mechanisms



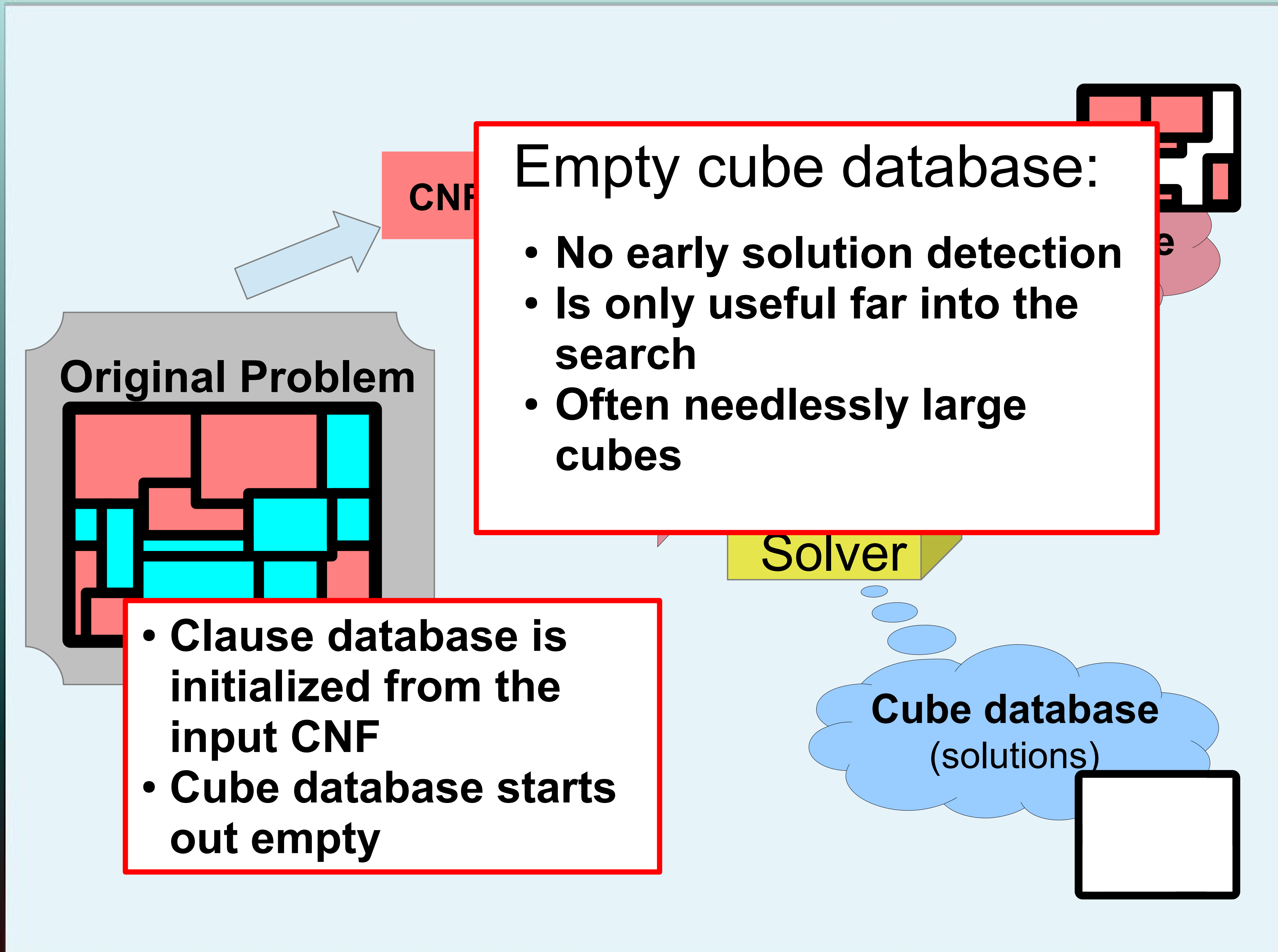
CNF
Original Problem
Clause database
(conflicts)
QBF
Solver
Cube database
(solutions)



- Clause database is initialized from the input CNF



- Clause database is initialized from the input CNF
- Cube database starts out empty


Empty cube database:
• No early solution detection
• Is only useful far into the search
• Often needlessly large cubes
CNF
Original Problem
Solver
Cube database
(solutions)
• Clause database is initialized from the input CNF
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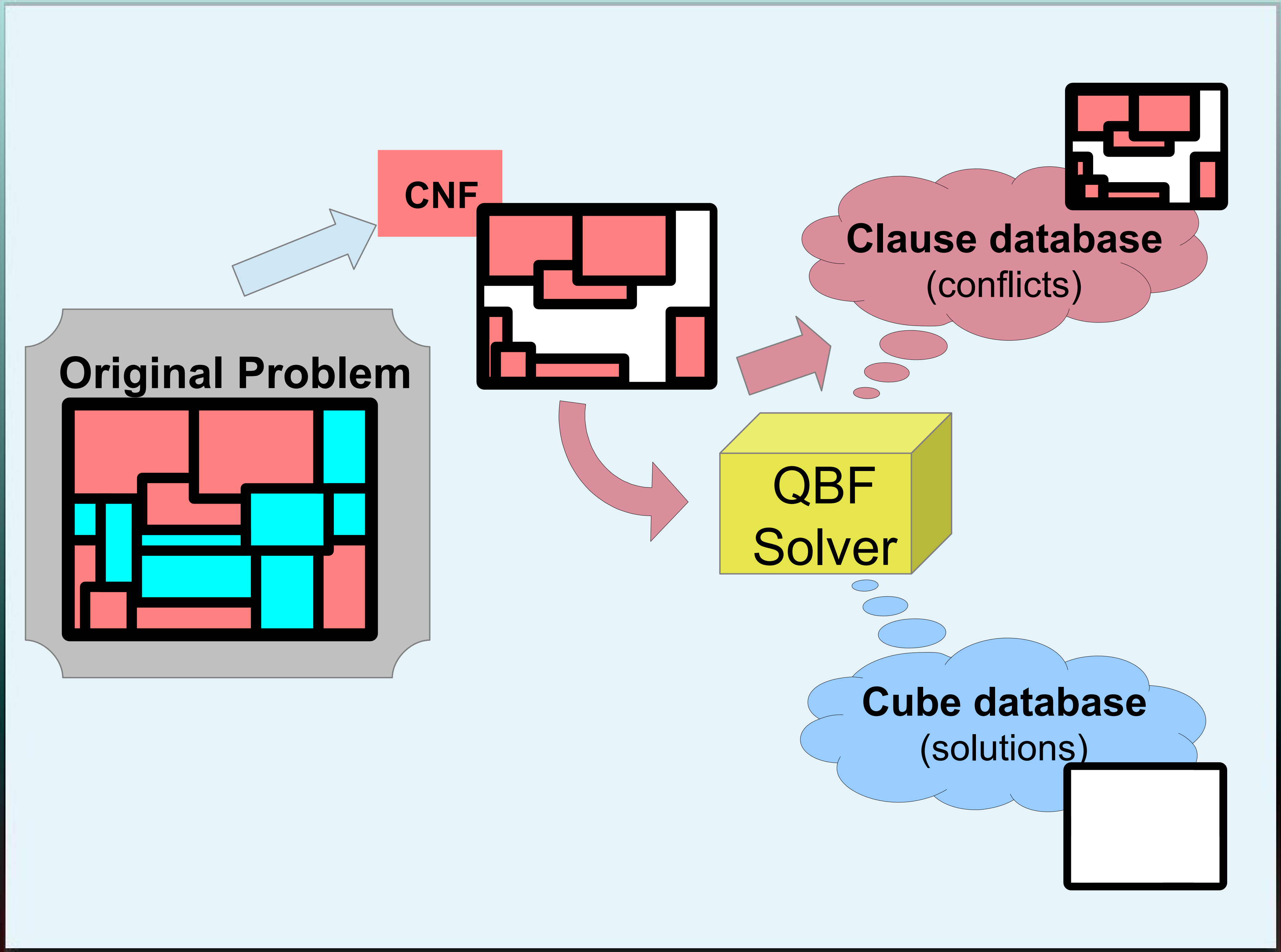
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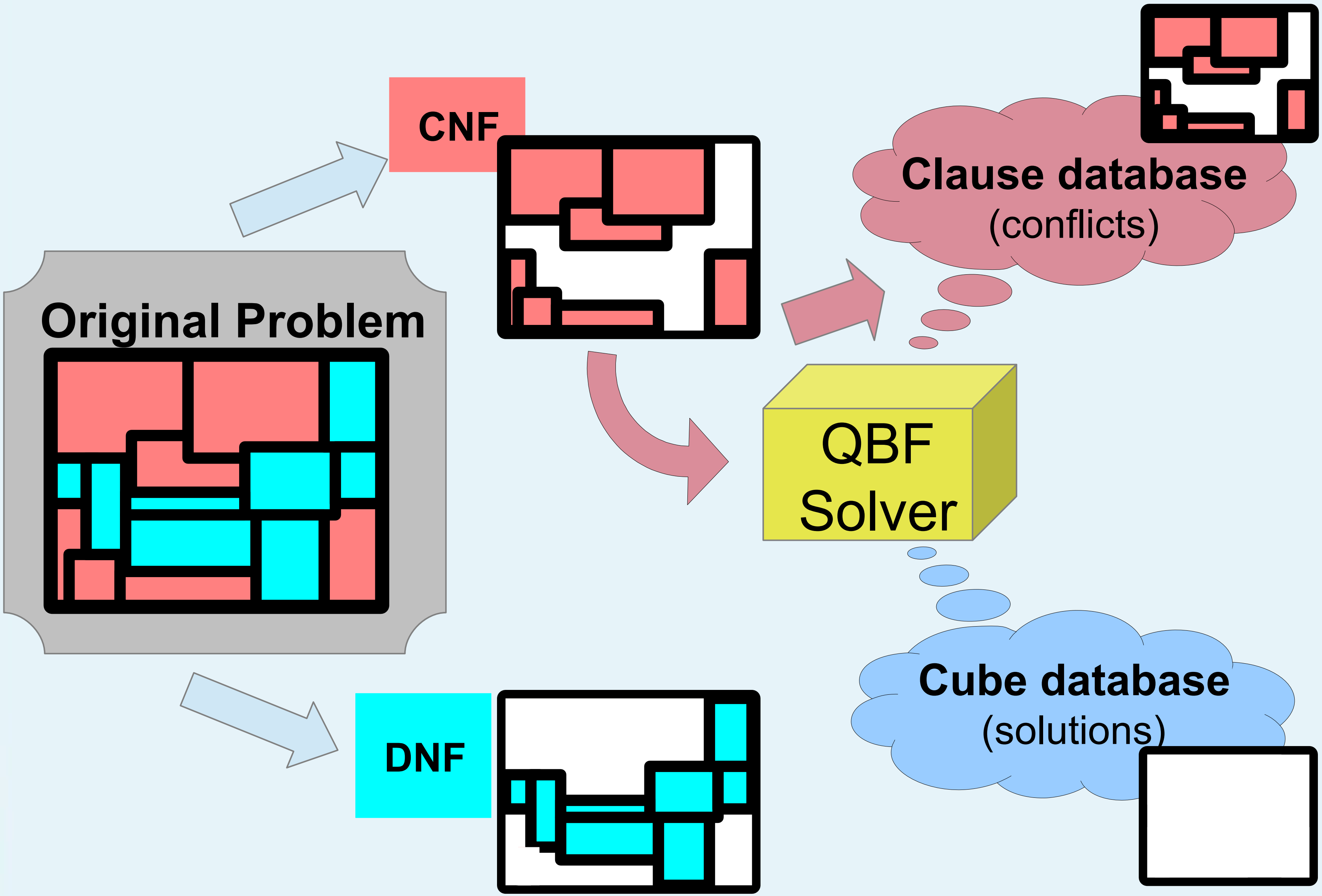
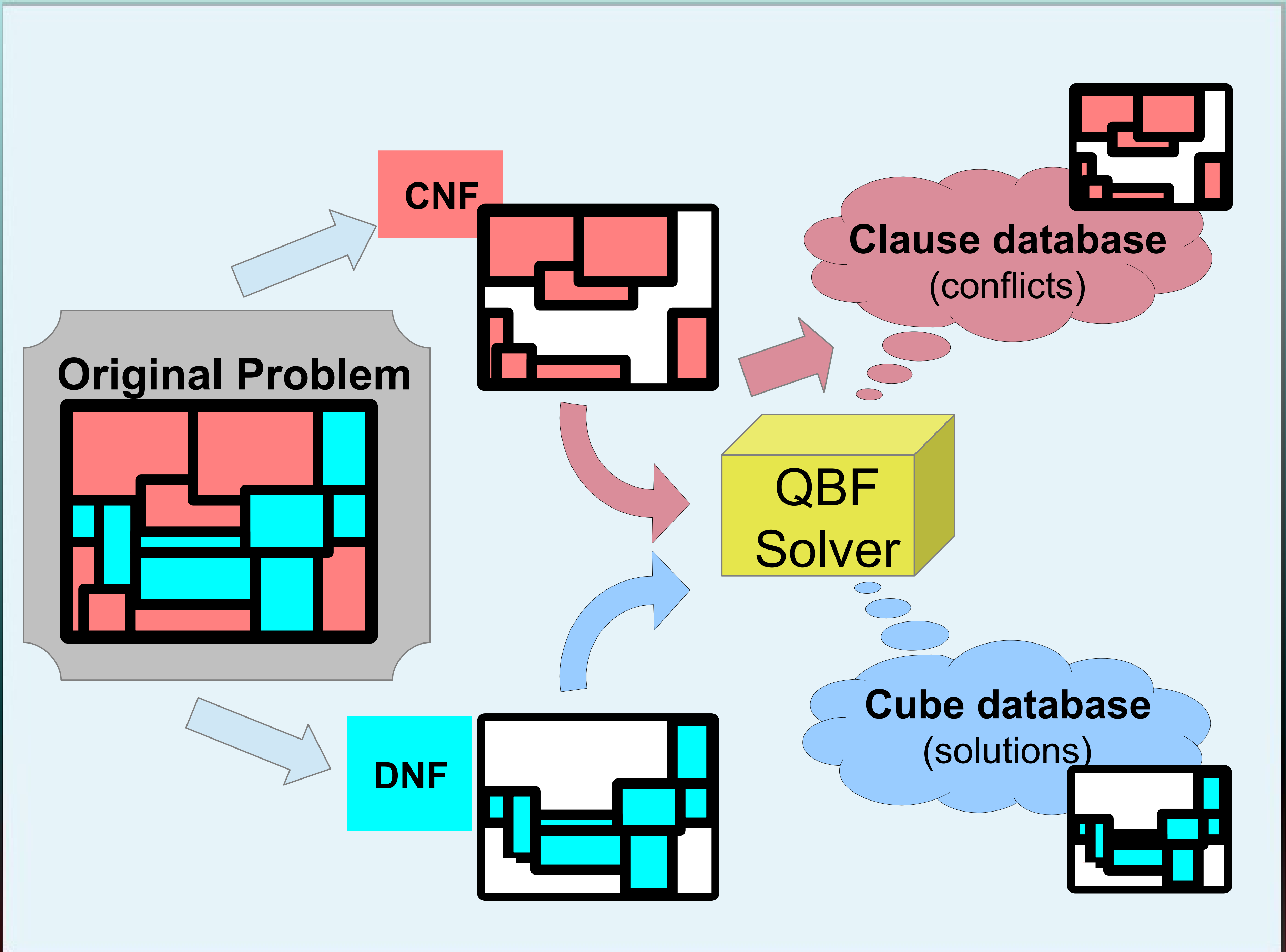


Original Problem
CNF
DNF
QBF Solver
Clause database
(conflicts)
Cube database
(solutions)



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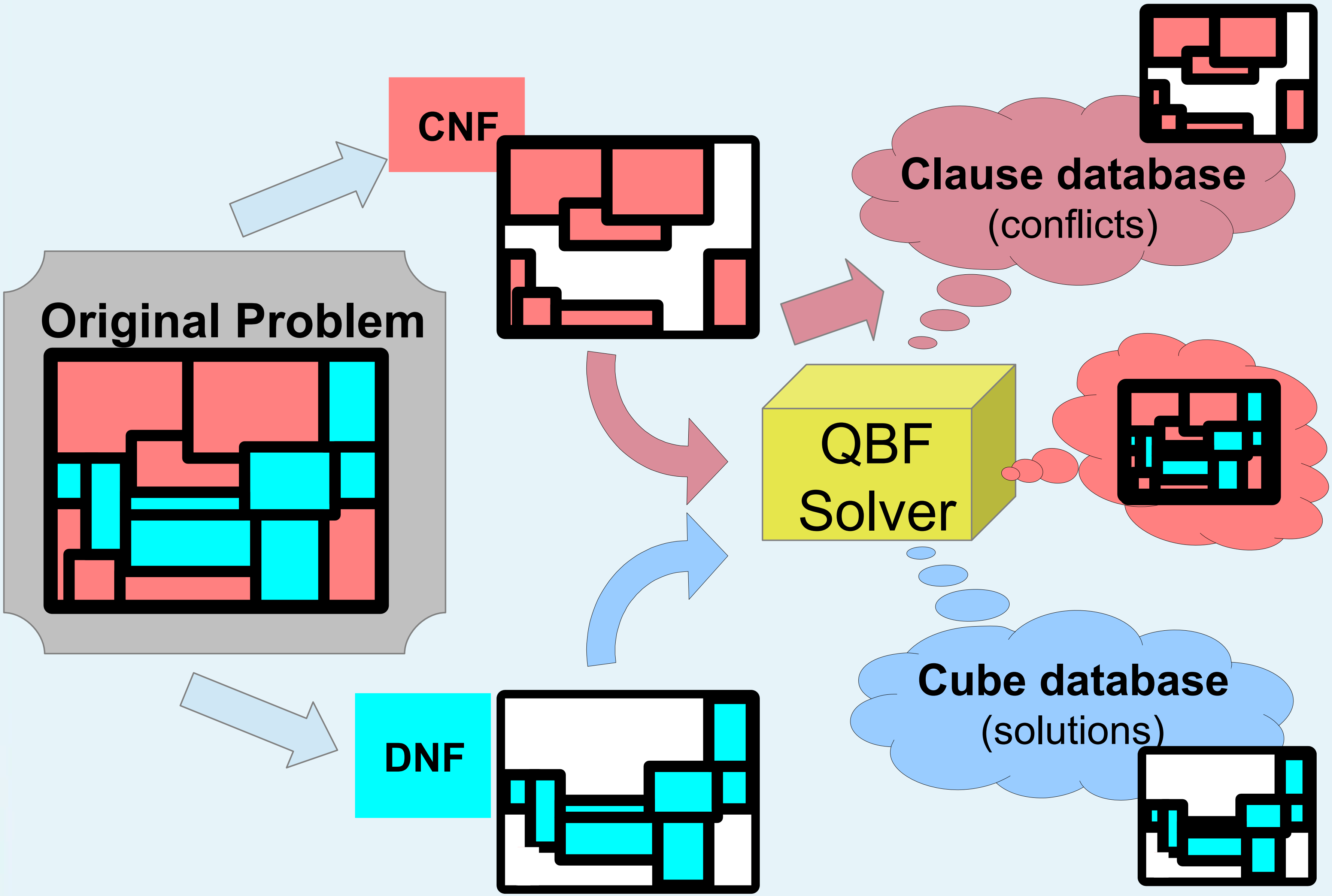
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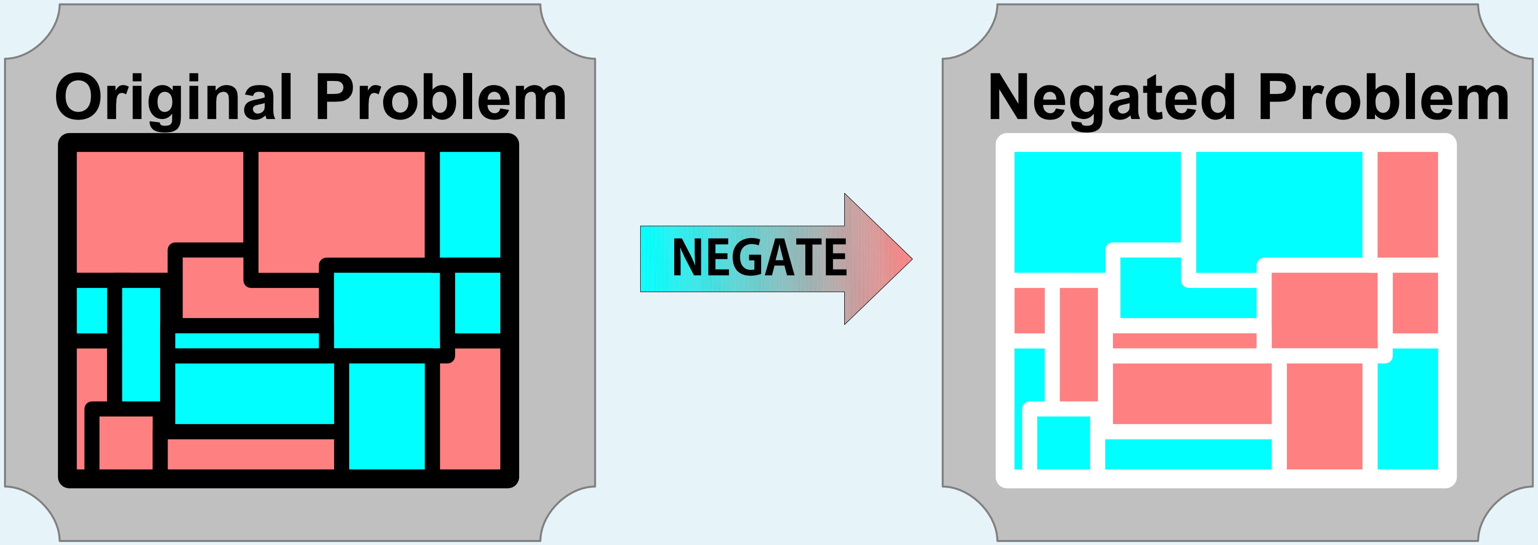
Getting solution structure



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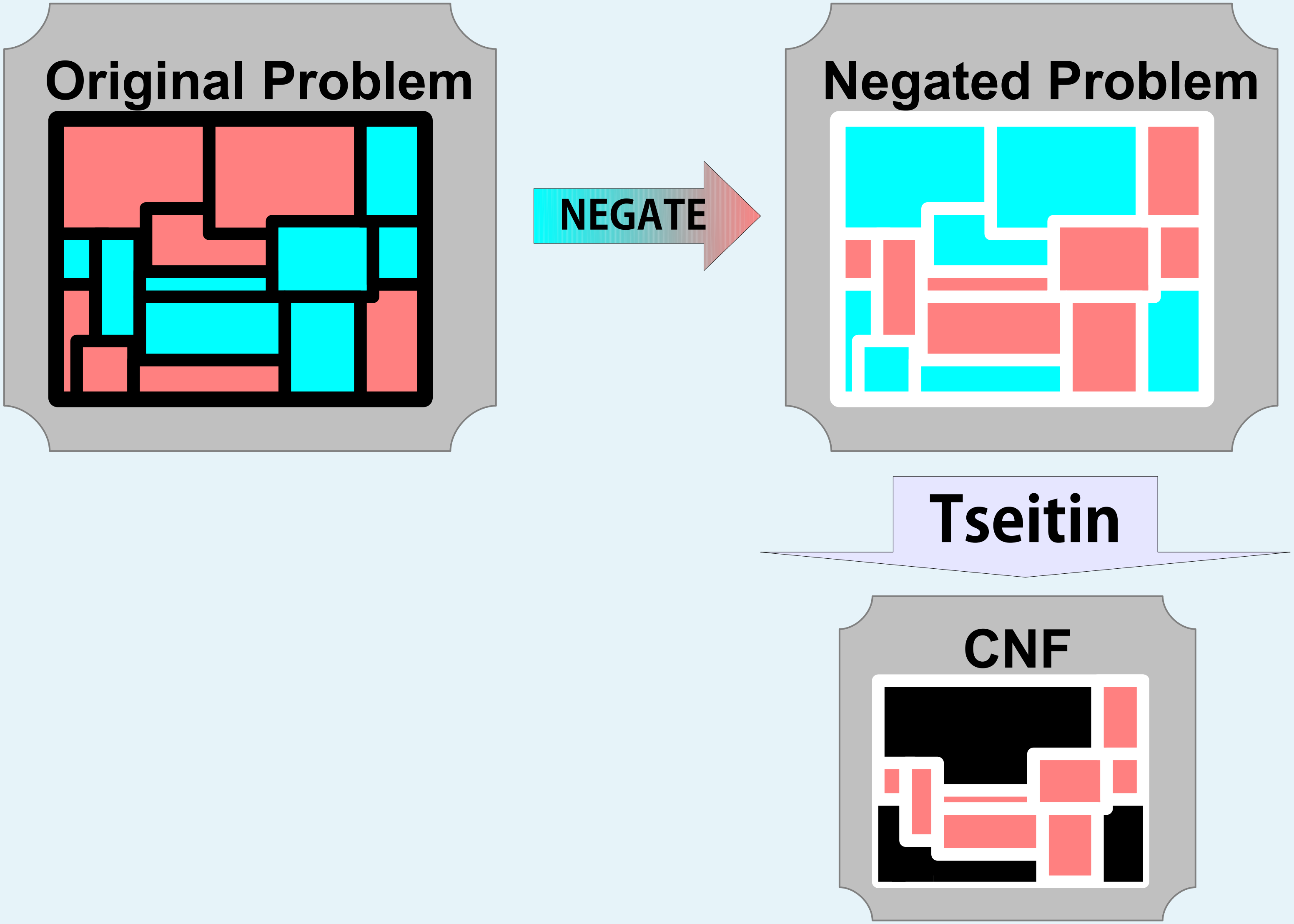

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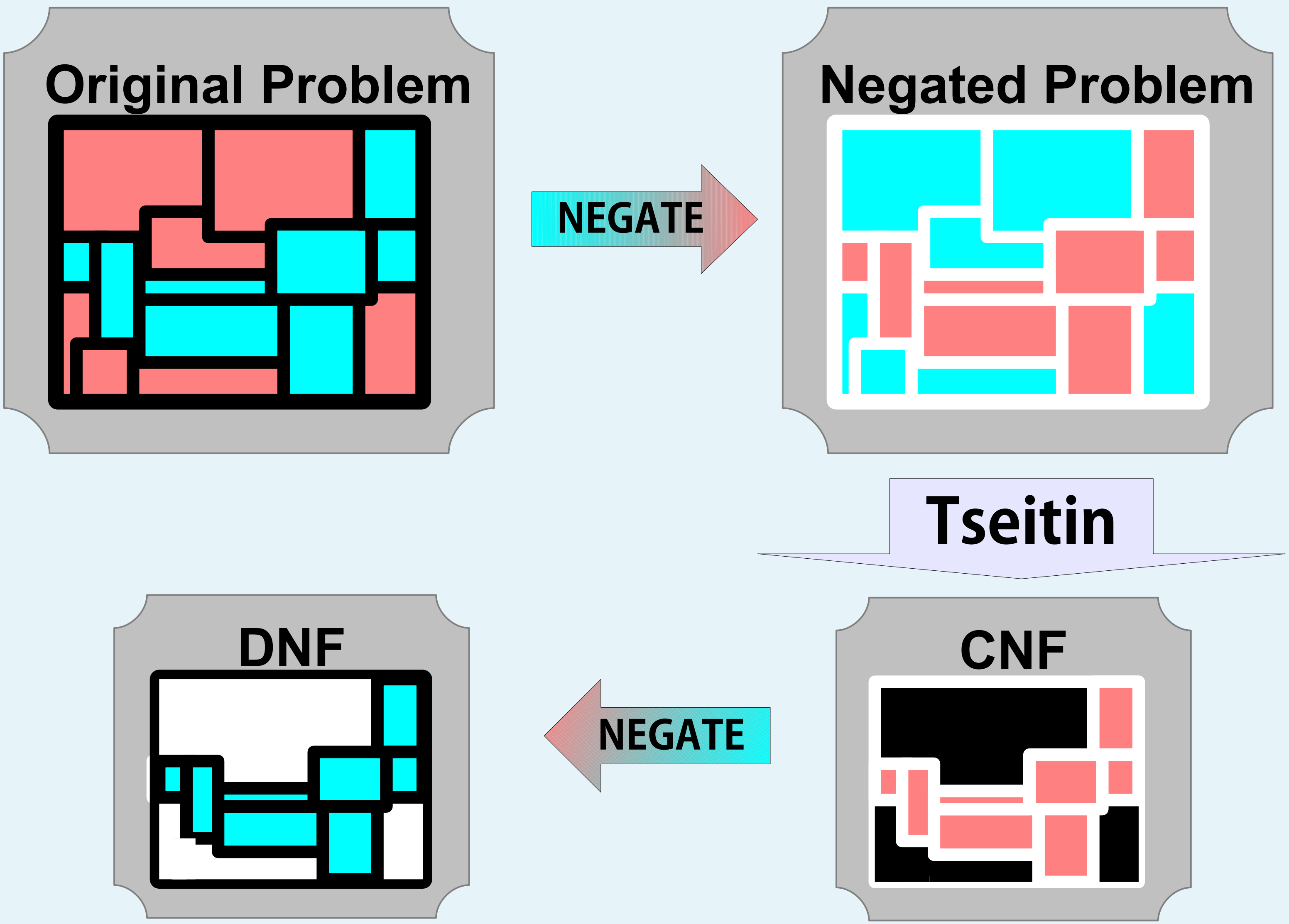

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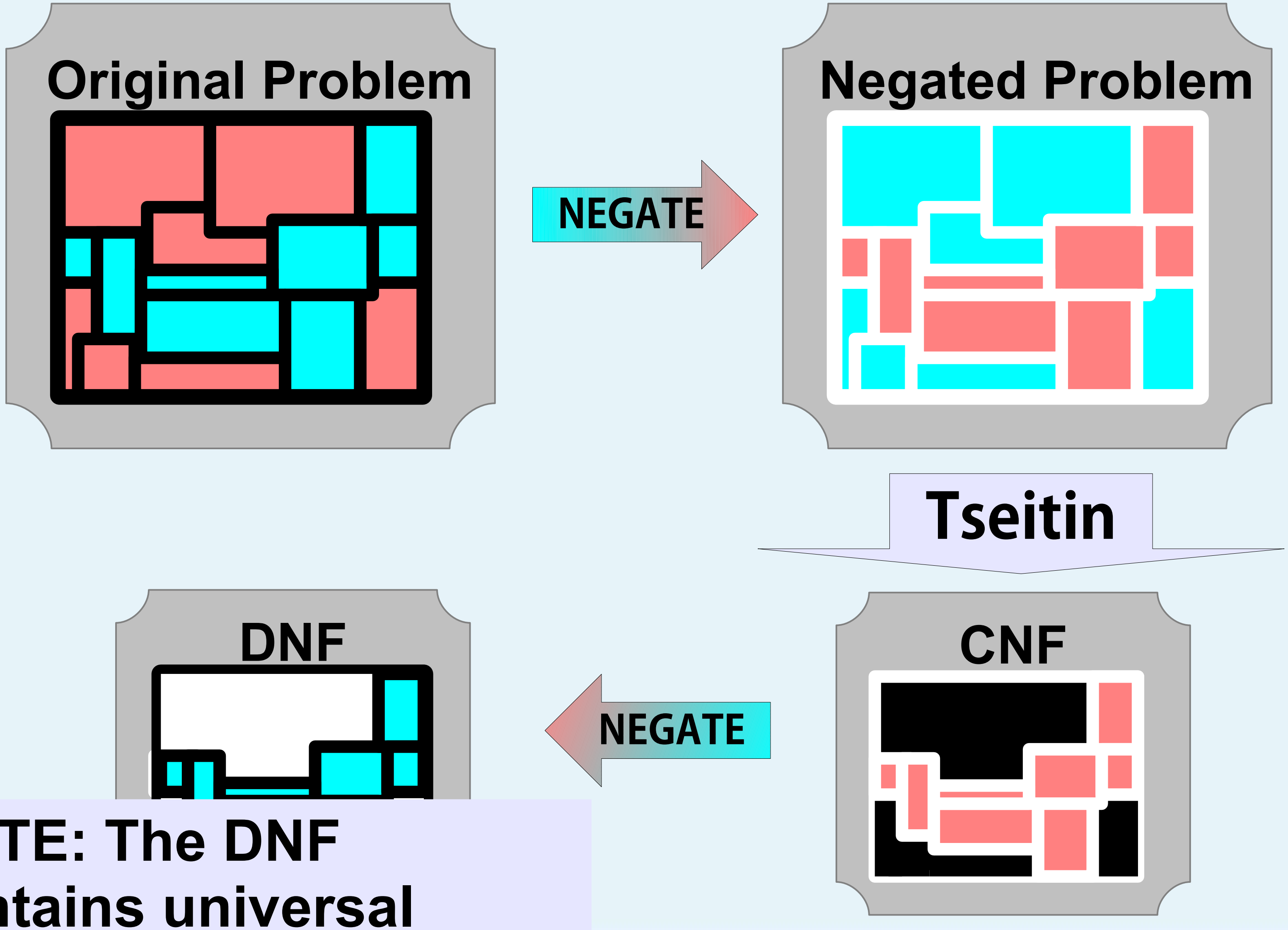
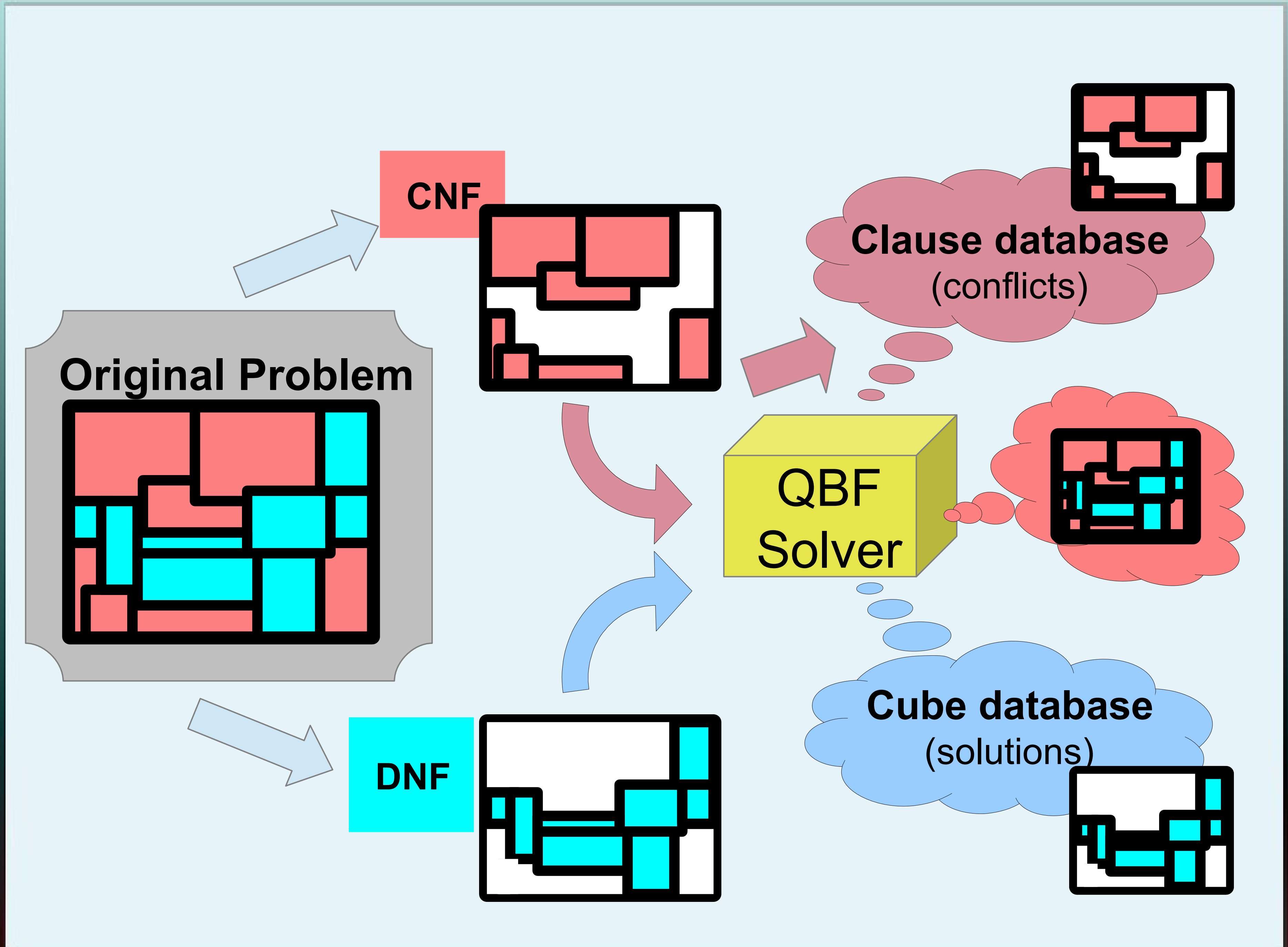


NOTE: The DNF contains universal Tseitin variables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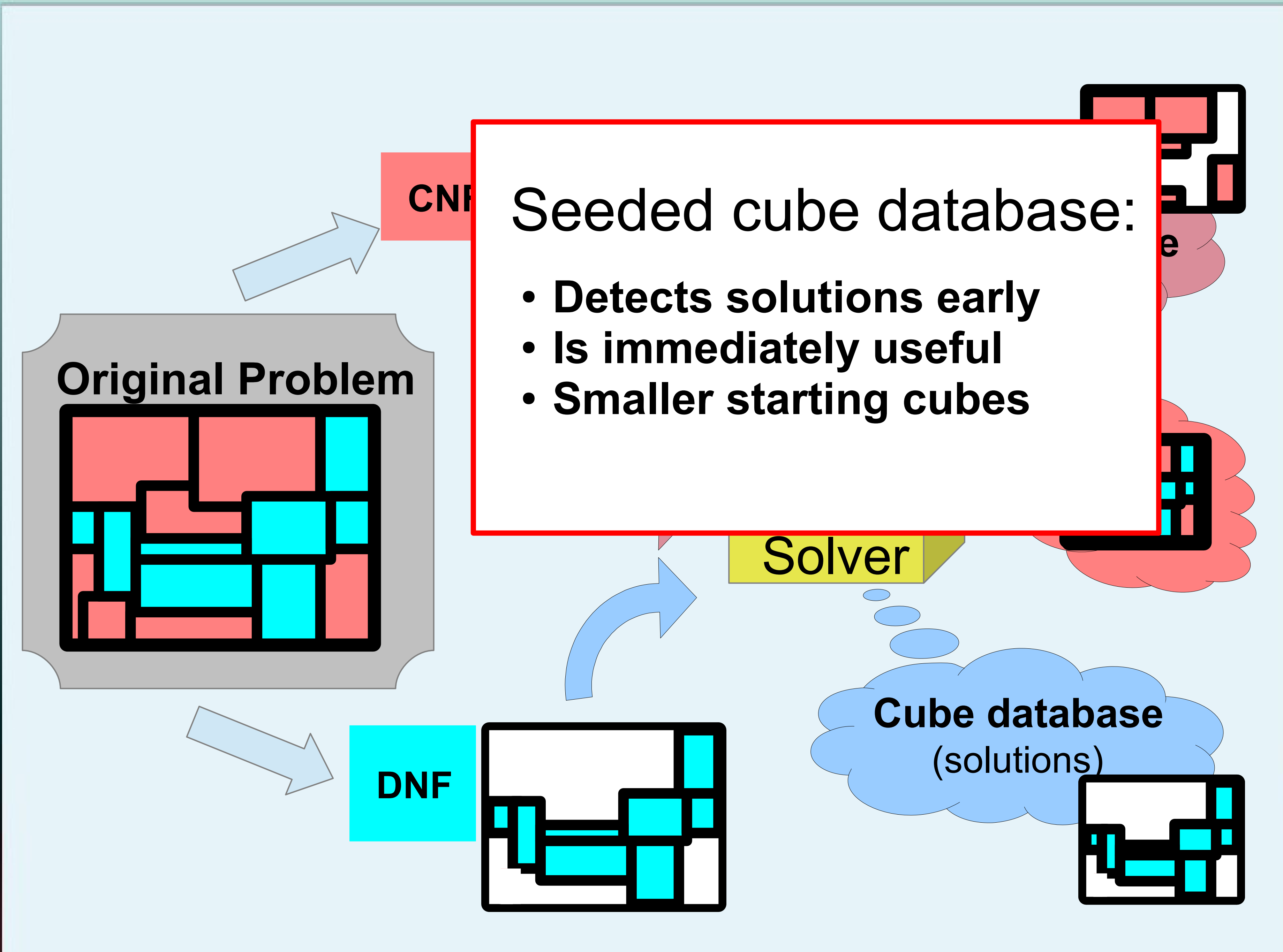

Seeded cube database:
• Detects solutions early
• Is immediately useful
• Smaller starting cubes
Original Problem
DNF
Solver
Cube database
(solutions)

Reconstructing structure

- Problem: original non-CNF is not always available
- Reconstruction methods exist, but they are necessarily unreliable and incomplete
- Want to take advantage of partially reconstructed information

Reconstructing structure

$$
\begin{aligned}
&(a \vee b \vee c) \\
\wedge &(e \vee f) \\
\wedge &(\neg a \vee \neg f \vee c) \\
\wedge &(a \vee d \vee e) \\
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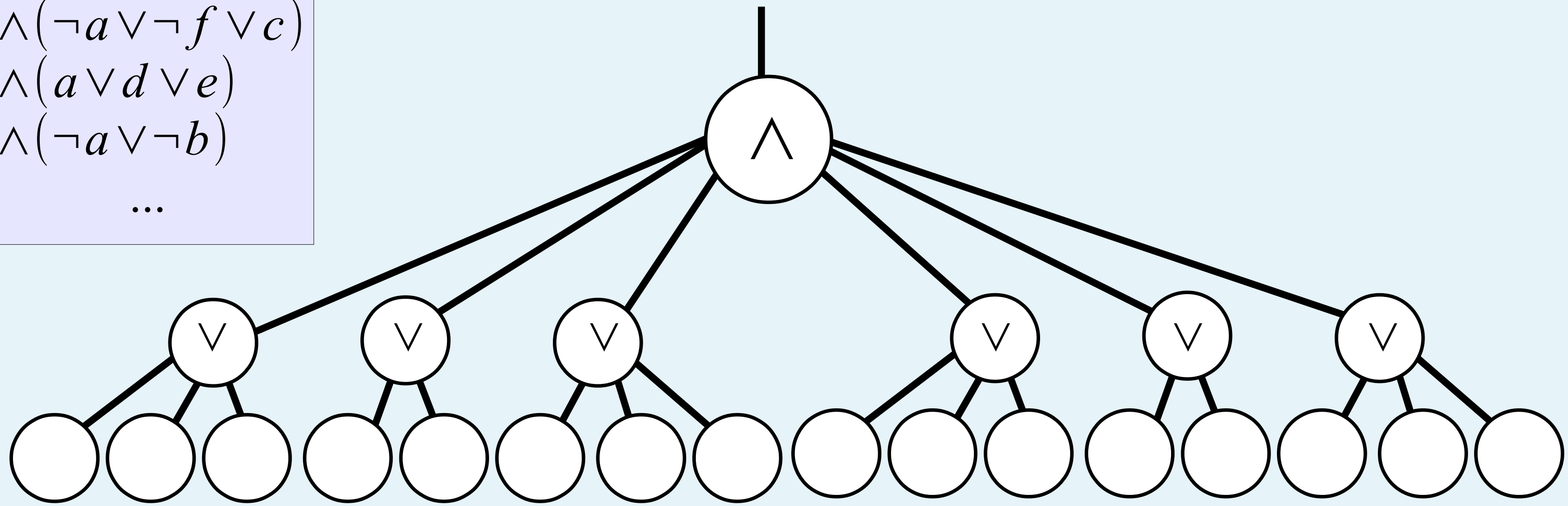
&\ldots
\end{aligned}
$$



- **CNF can be viewed as a flat tree**

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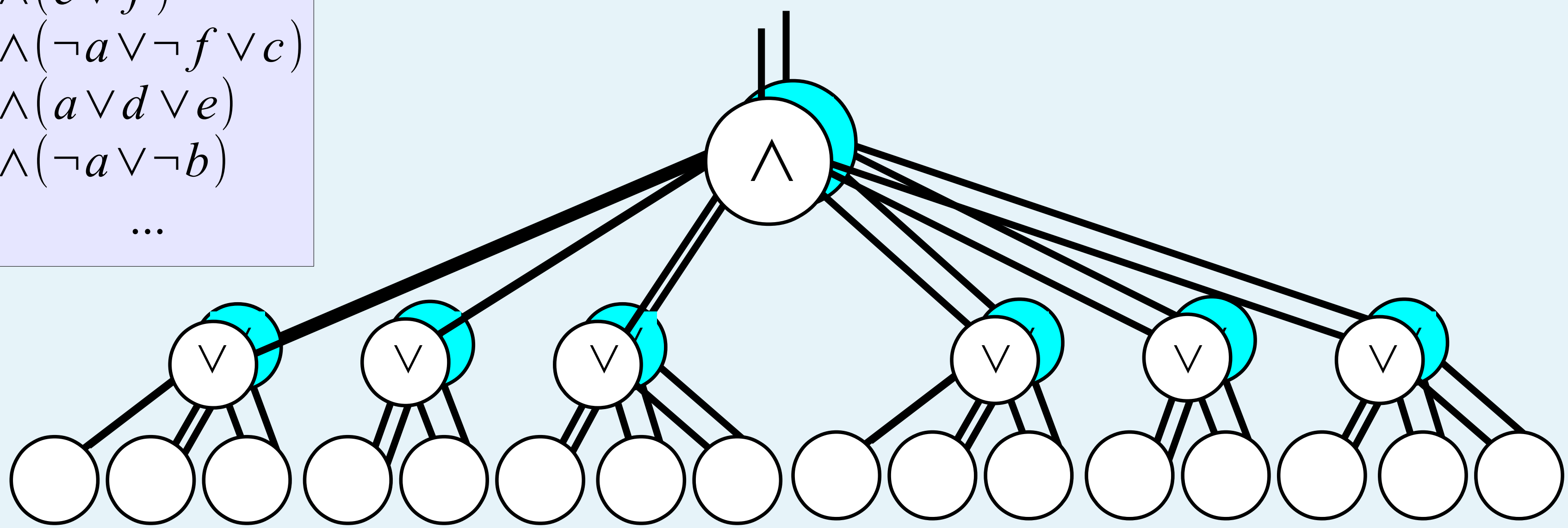

- **CNF can be viewed as a flat tree**
- **Negating it and converting to DNF would create a new variable for every clause**

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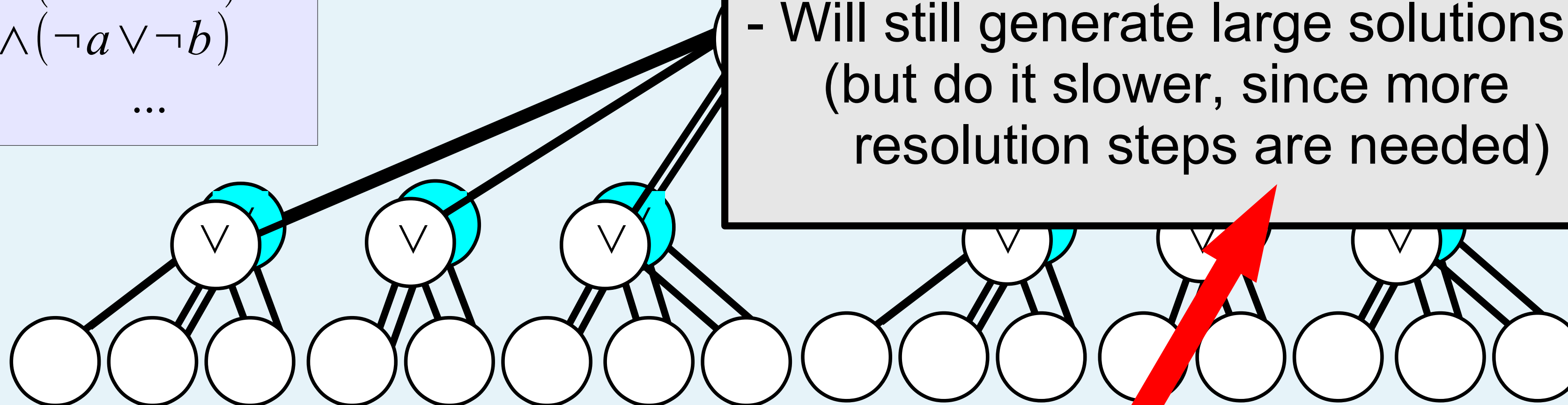
&\quad \ldots
\end{aligned}
$$

Is not generally useful:
- Will not help early detection
- Will not help propagation
- Will still generate large solutions
  (but do it slower, since more
  resolution steps are needed)



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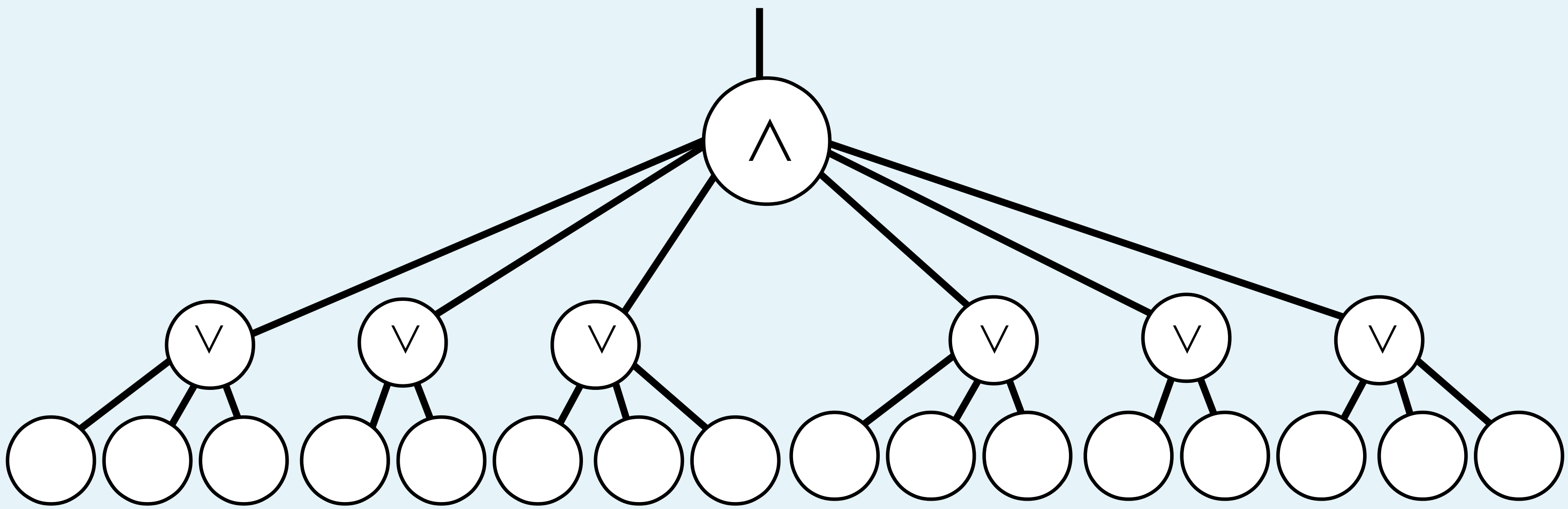

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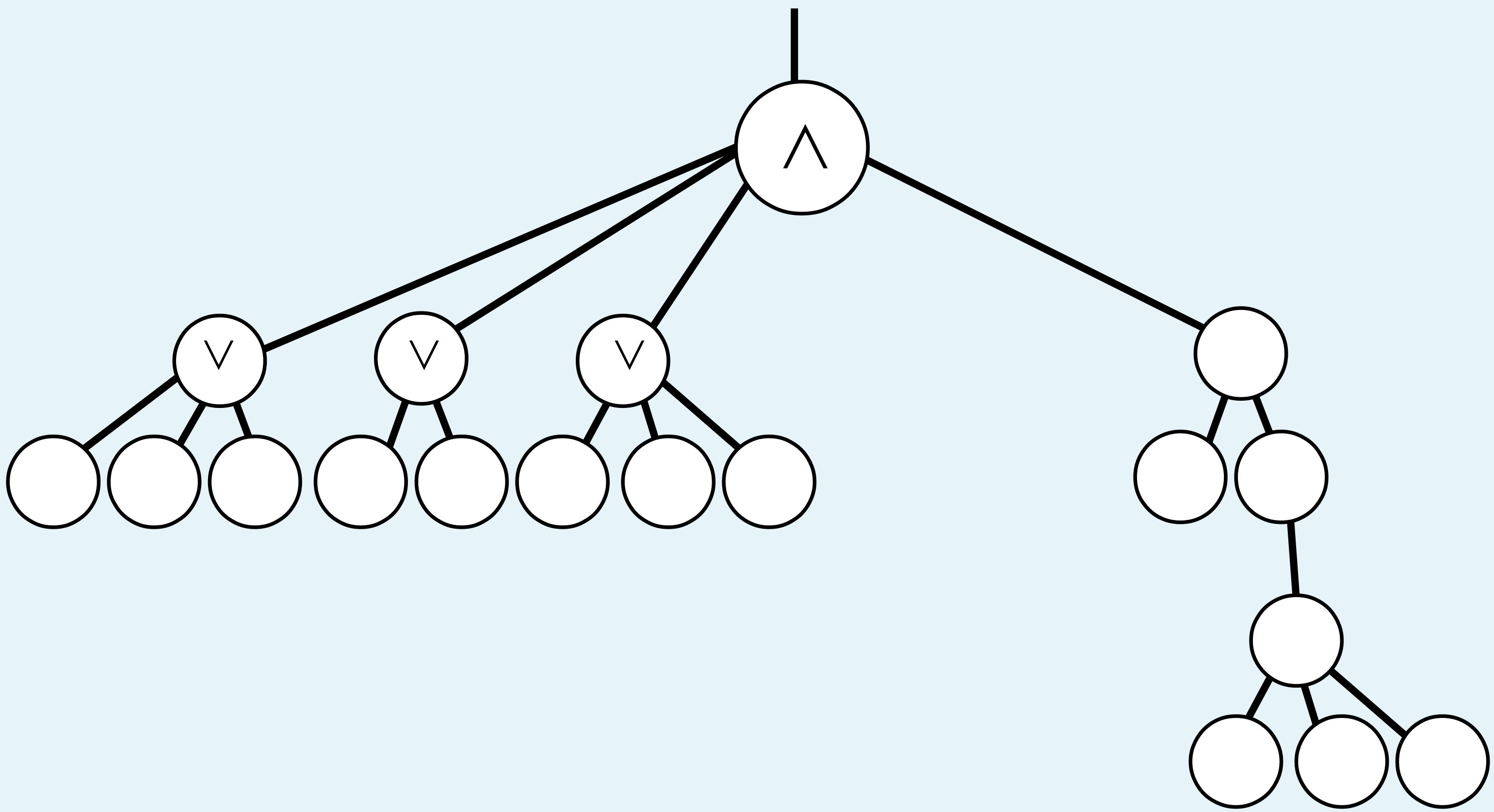

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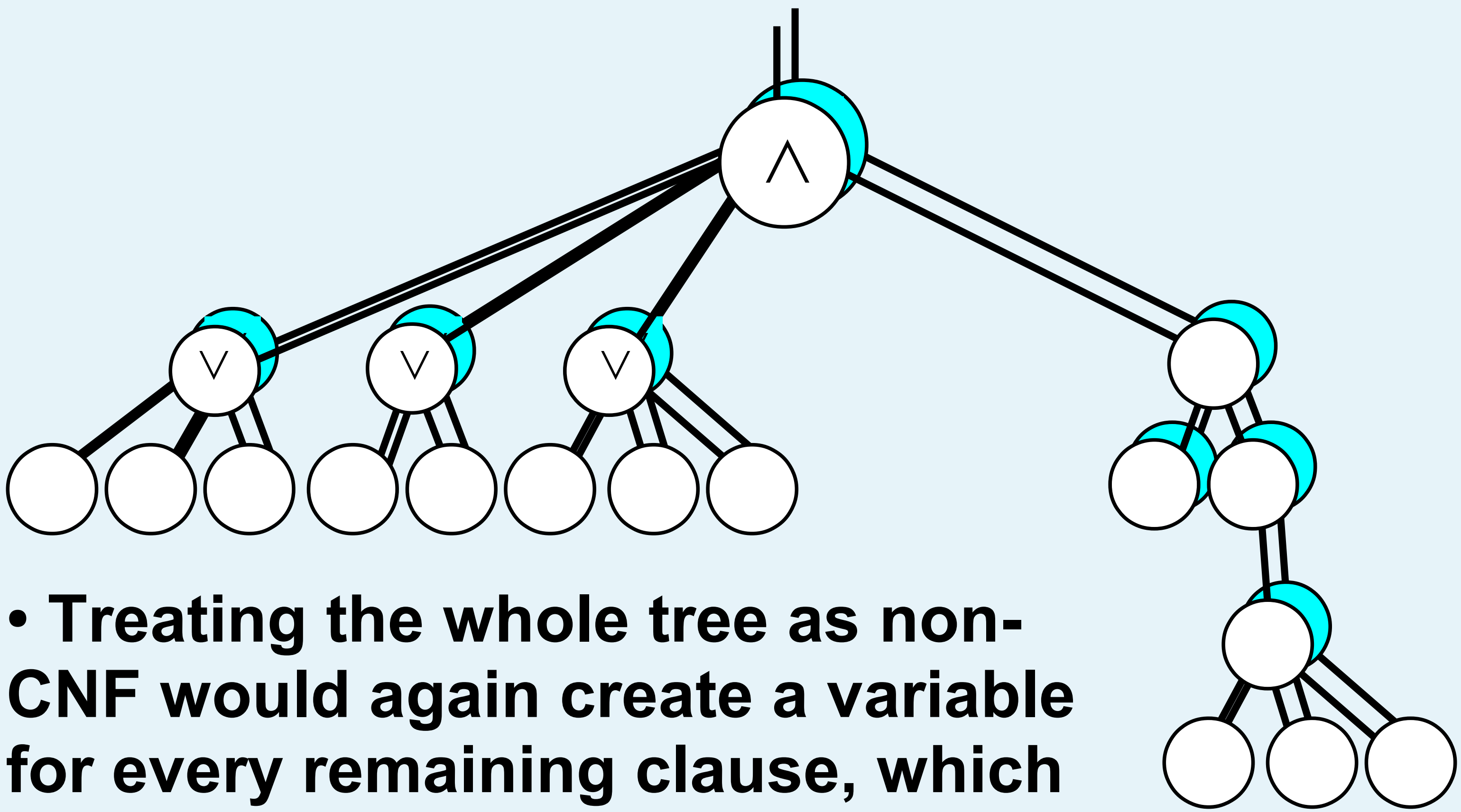
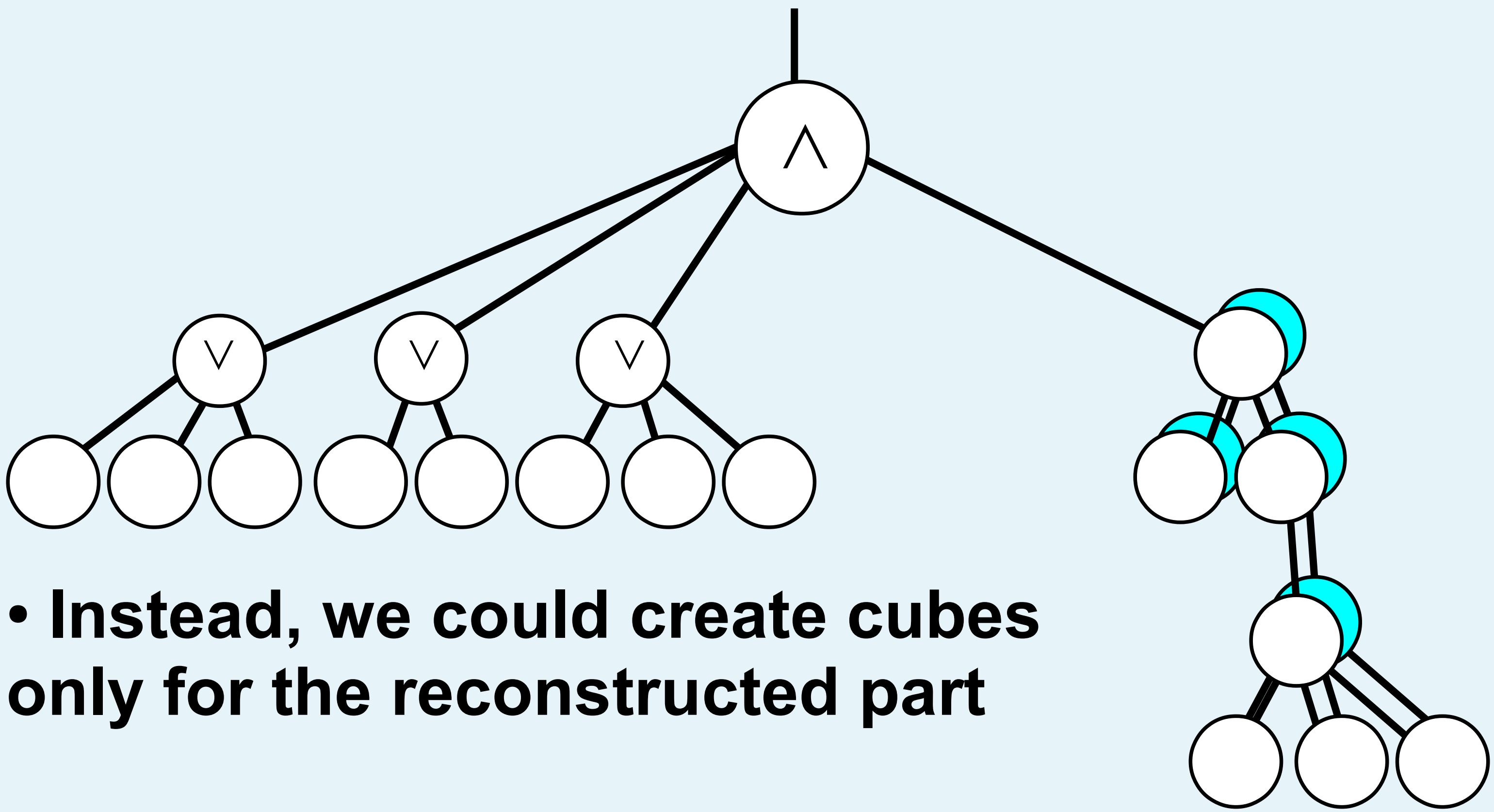


- **Treating the whole tree as non-CNF would again create a variable for every remaining clause, which is inefficient**

Reconstructing structure



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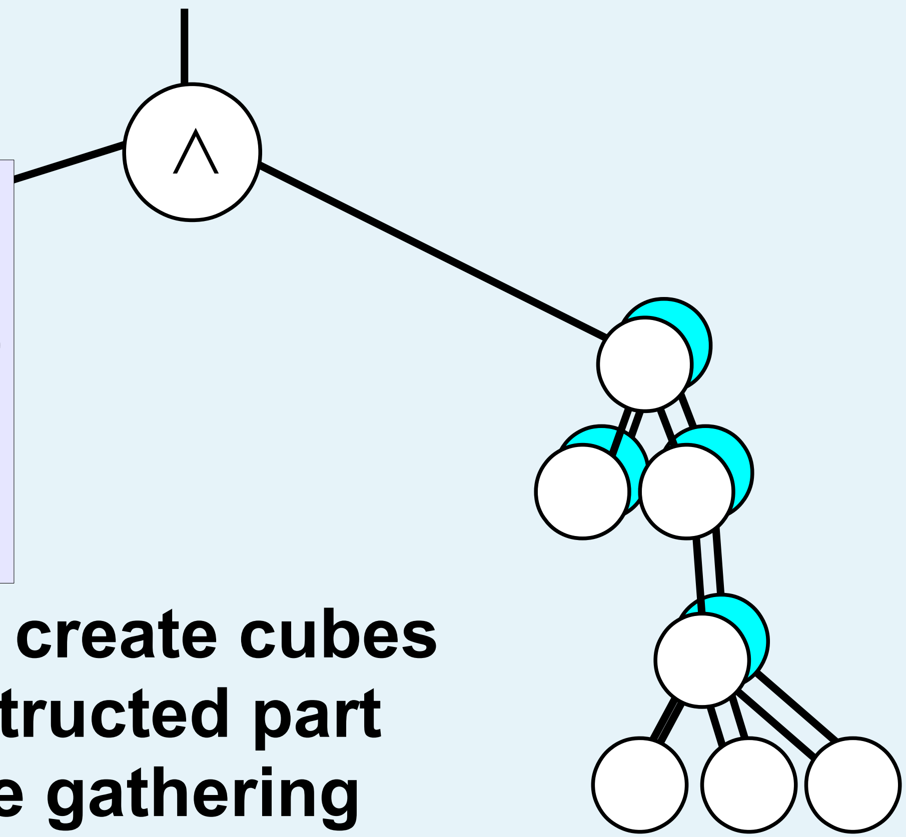

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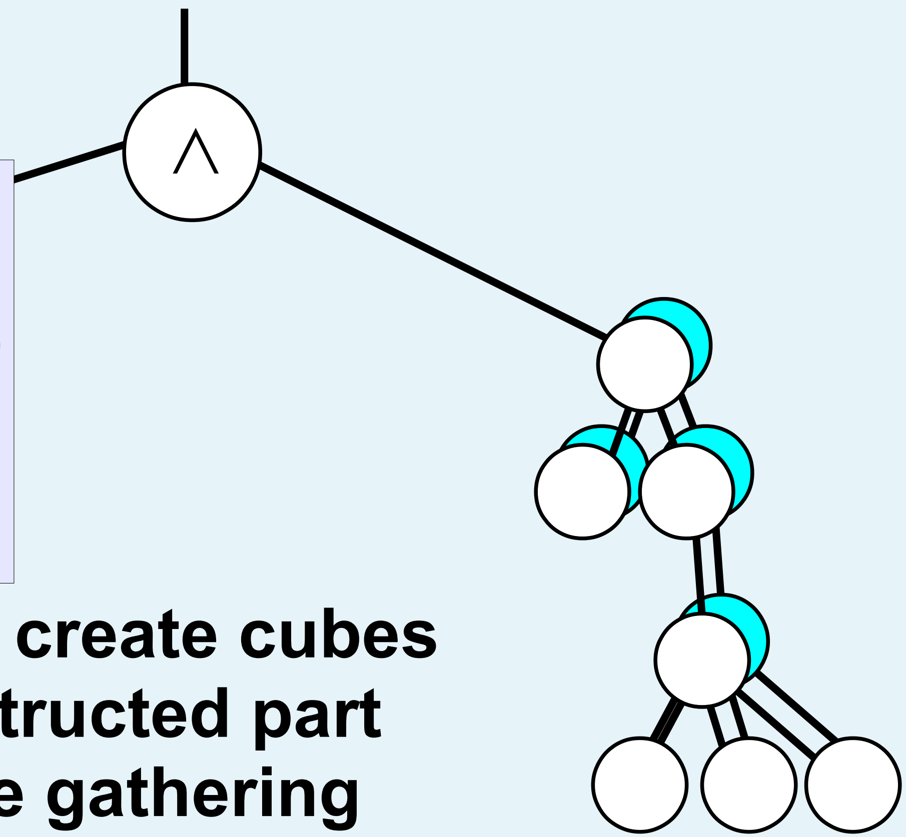
- **Let the usual cube gathering happen in the remainder**

Reconstructing structure



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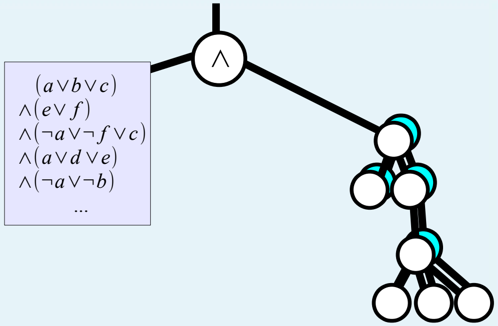
Partial Duality



- Will not gather cubes from definition clauses
- Creates new universal variables to make cubes more expressive
- No efficiency loss on poorly reconstructed instances
- Complete dual propagation on fully reconstructed instances

Plaisted-Greenbaum

- Instead of equivalences, uses implication for variable definitions
- Can be reconstructed using simple syntactic properties

$(\alpha_1 \vee x)$	$(\beta_1 \vee \neg x)$
$(\alpha_2 \vee x)$	$(\beta_2 \vee \neg x)$
$(\alpha_3 \vee x)$	$(\beta_3 \vee \neg x)$
$(\alpha_4 \vee x)$	$(\beta_4 \vee \neg x)$
...	...

$$\begin{array}{ll} (\alpha_1 \vee x) & (\beta_1 \vee \neg x) \\ (\alpha_2 \vee x) & (\beta_2 \vee \neg x) \\ (\alpha_3 \vee x) & (\beta_3 \vee \neg x) \\ (\alpha_4 \vee x) & (\beta_4 \vee \neg x) \\ \ldots & \ldots \end{array}$$

x is tailing

$$\underbrace{\begin{array}{c}(\alpha_1 \vee x)\\(\alpha_2 \vee x)\\(\alpha_3 \vee x)\\(\alpha_4 \vee x)\\ \ldots\end{array}}_{x \text{ is tailing}} \qquad \begin{array}{c}(\beta_1 \vee \neg x)\\(\beta_2 \vee \neg x)\\(\beta_3 \vee \neg x)\\(\beta_4 \vee \neg x)\\ \ldots\end{array}$$

Then: $(\neg\alpha_1 \vee \neg\alpha_2 \vee \neg\alpha_3 \vee \ldots) \rightarrow x$

$$(\alpha_1 \vee x) \qquad (\beta_1 \vee \neg x)$$
$$(\alpha_2 \vee x) \qquad (\beta_2 \vee \neg x)$$
$$(\alpha_3 \vee x) \qquad (\beta_3 \vee \neg x)$$
$$(\alpha_4 \vee x) \qquad (\beta_4 \vee \neg x)$$
$$\underbrace{\quad \ldots \quad}_{x \text{ is tailing}} \qquad \ldots$$

Then: $(\neg\alpha_1 \vee \neg\alpha_2 \vee \neg\alpha_3 \vee \ldots) \rightarrow x$

Set the dual for x to be a new universal u such that:

$$(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \ldots) \rightarrow \neg u$$

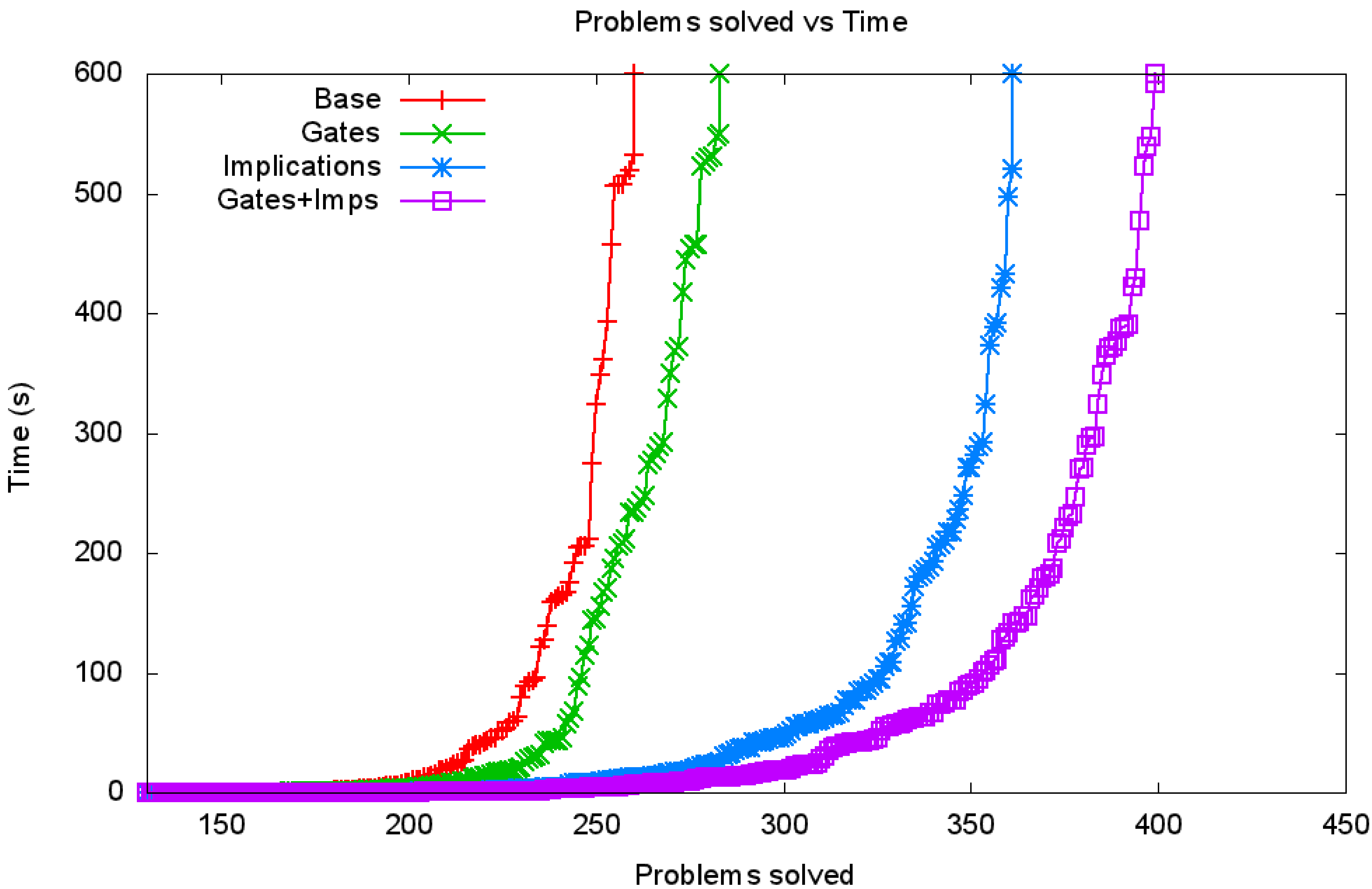
$$\underbrace{\begin{array}{c} (\alpha_1 \vee x) \\ (\alpha_2 \vee x) \\ (\alpha_3 \vee x) \\ (\alpha_4 \vee x) \\ \ldots \end{array}}_{x \text{ is tailing}} \qquad\qquad \begin{array}{c} (\beta_1 \vee \neg x) \\ (\beta_2 \vee \neg x) \\ (\beta_3 \vee \neg x) \\ (\beta_4 \vee \neg x) \\ \ldots \end{array}$$

Then: $(\neg\alpha_1 \vee \neg\alpha_2 \vee \neg\alpha_3 \vee \ldots) \rightarrow x$

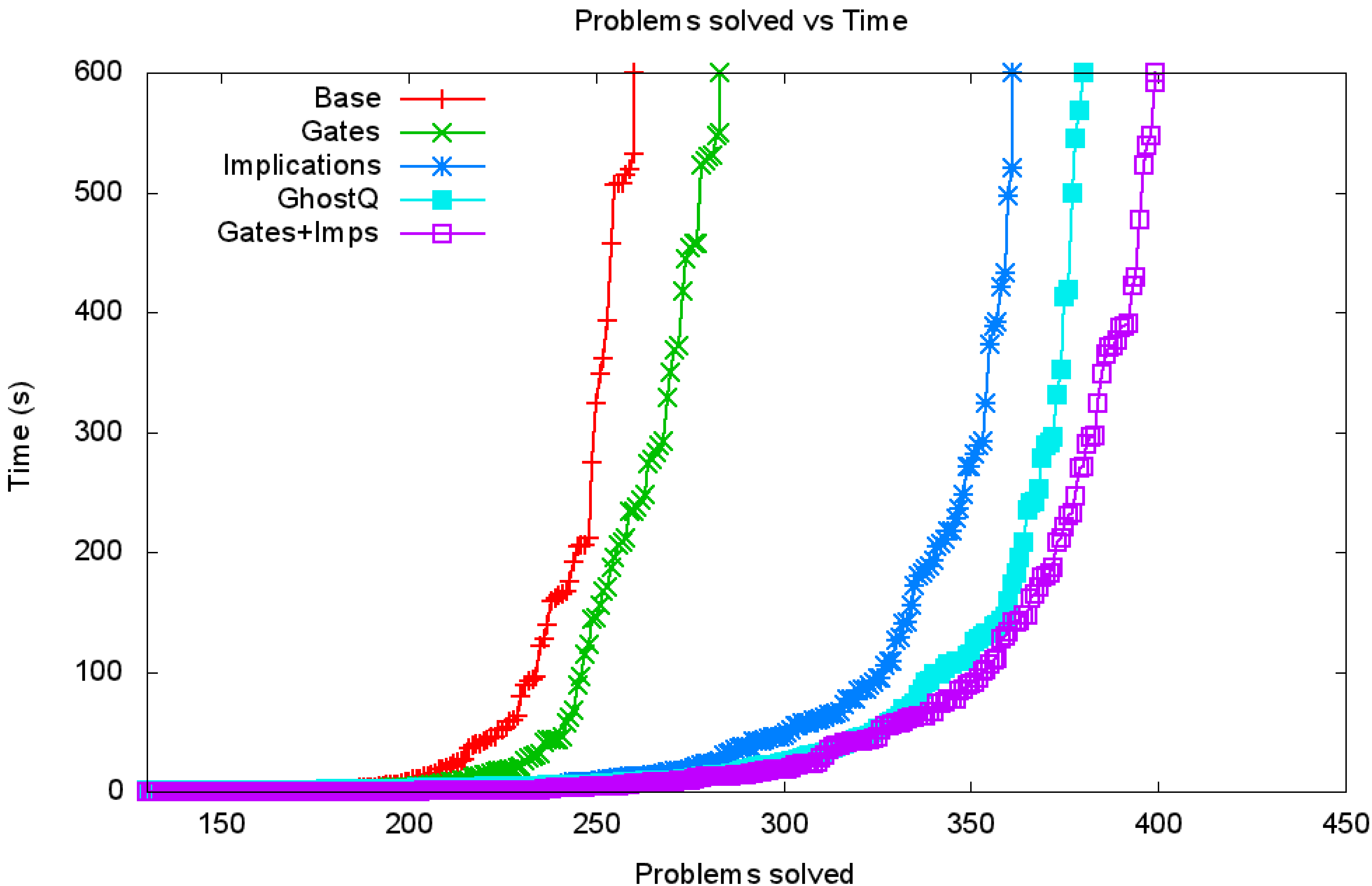
Set the dual for x **to be a new universal** u **such that:**

$$(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \ldots) \rightarrow \neg u$$

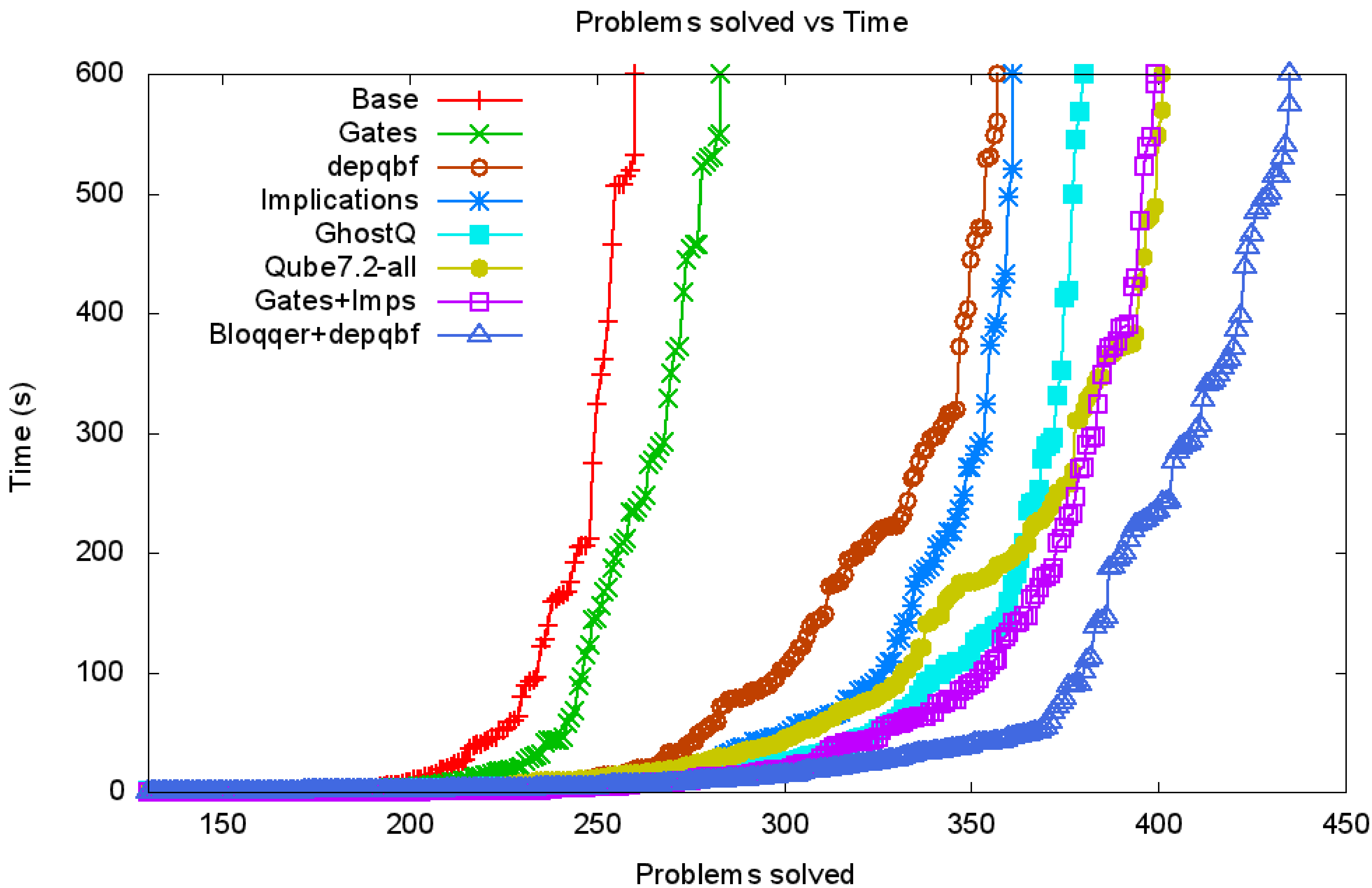
Intuitively: set $(\neg\alpha_1 \vee \neg\alpha_2 \vee \neg\alpha_3 \vee \ldots) \equiv x$
and then remove blocked clauses and cubes



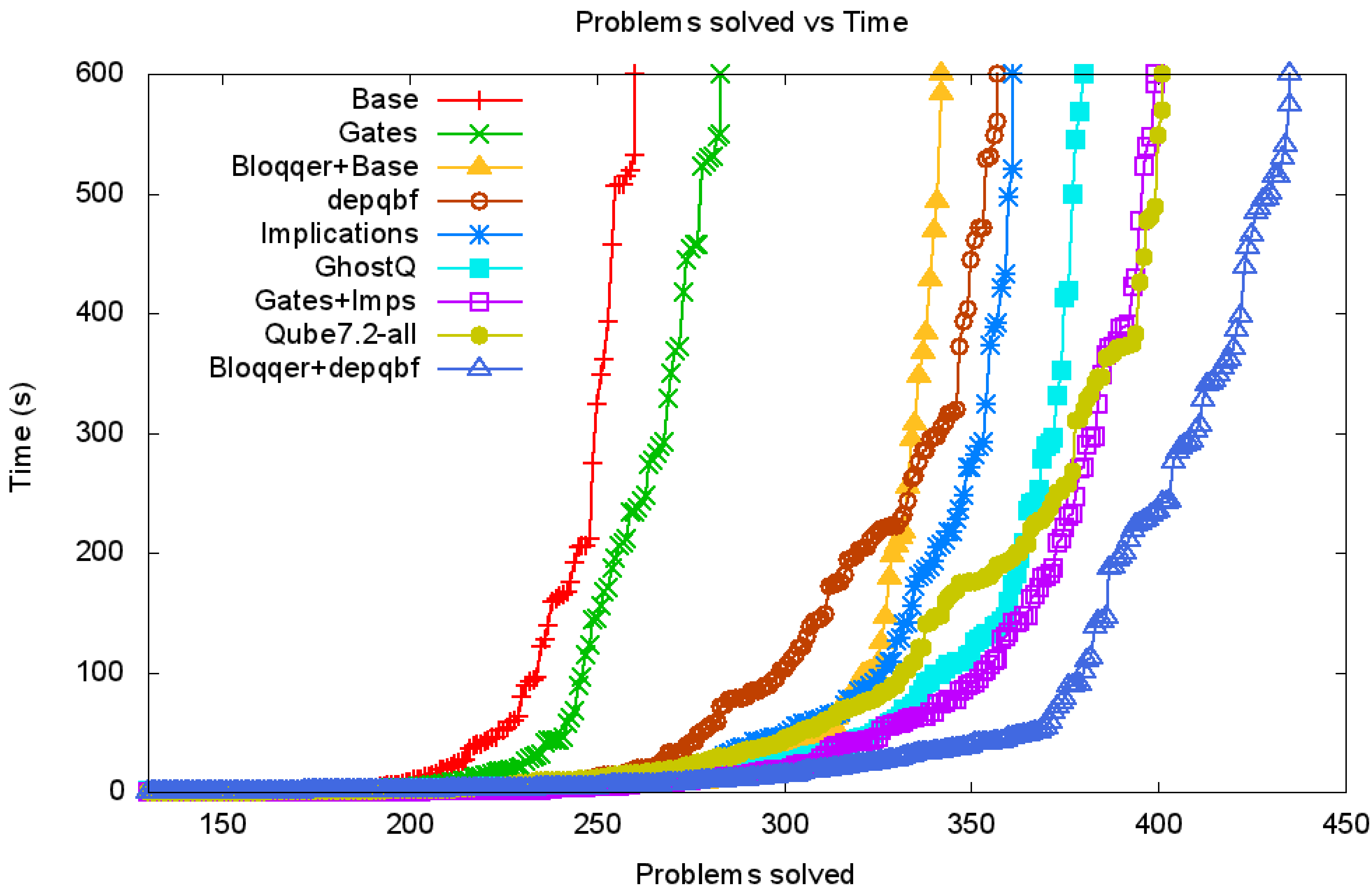
Problems solved vs Time
Base
Gates
Implications
Gates+Imps
Time (s)
0
100
200
300
400
500
600
150
200
250
300
350
400
450
Problems solved



Problems solved vs Time
Base
Gates
Implications
GhostQ
Gates+Imps
Time (s)
Problems solved
0
100
200
300
400
500
600
150
200
250
300
350
400
450



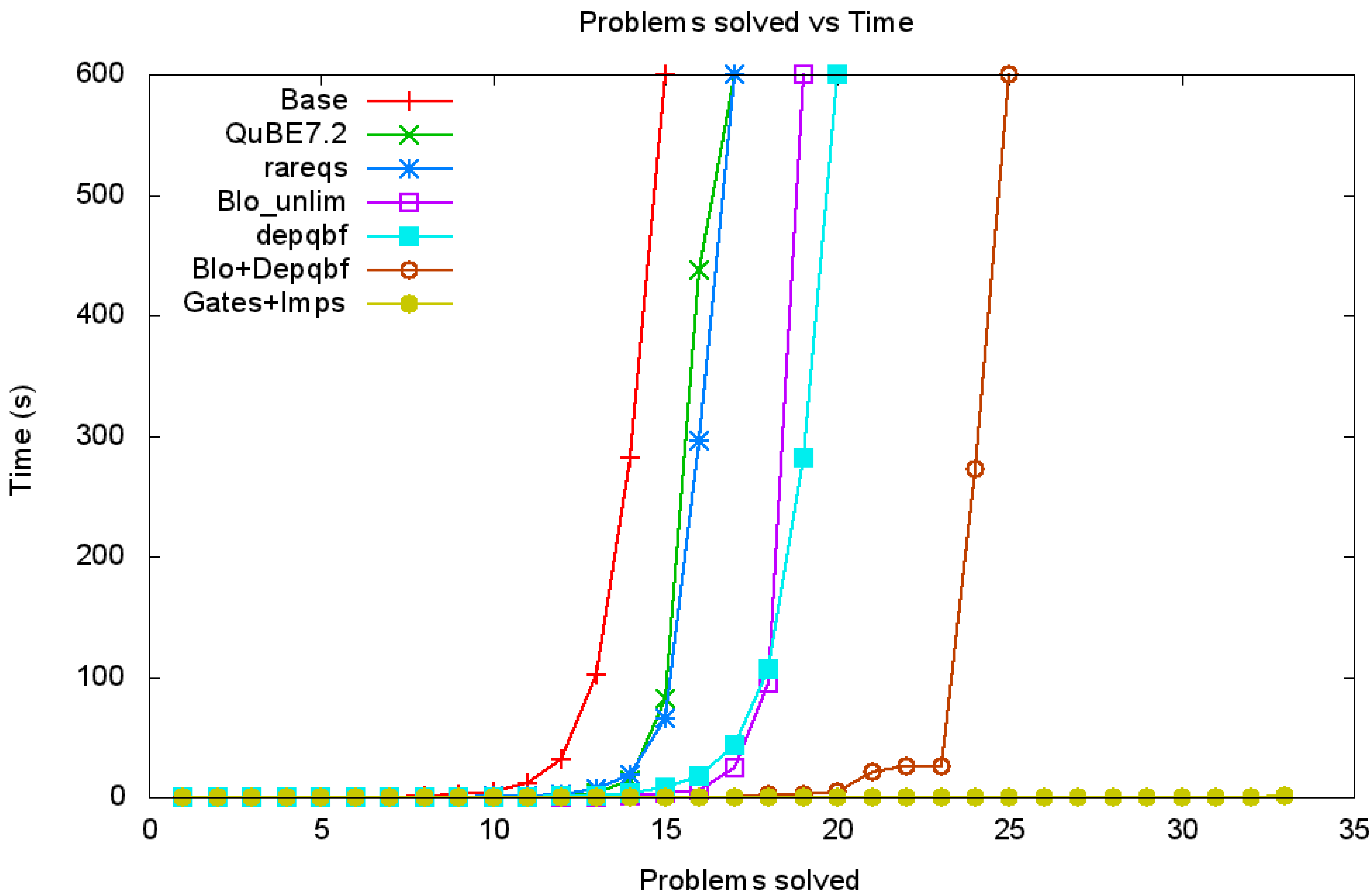
Problems solved vs Time
Time (s)
Problems solved
600
500
400
300
200
100
0
150
200
250
300
350
400
450
Base
Gates
depqbf
Implications
GhostQ
Qube7.2-all
Gates+Imps
Bloqqer+depqbf


Problems solved vs Time
Base
Gates
Bloqqer+Base
depqbf
Implications
GhostQ
Gates+Imps
Qube7.2-all
Bloqqer+depqbf
Time (s)
600
500
400
300
200
100
0
150
200
250
300
350
400
450
Problems solved

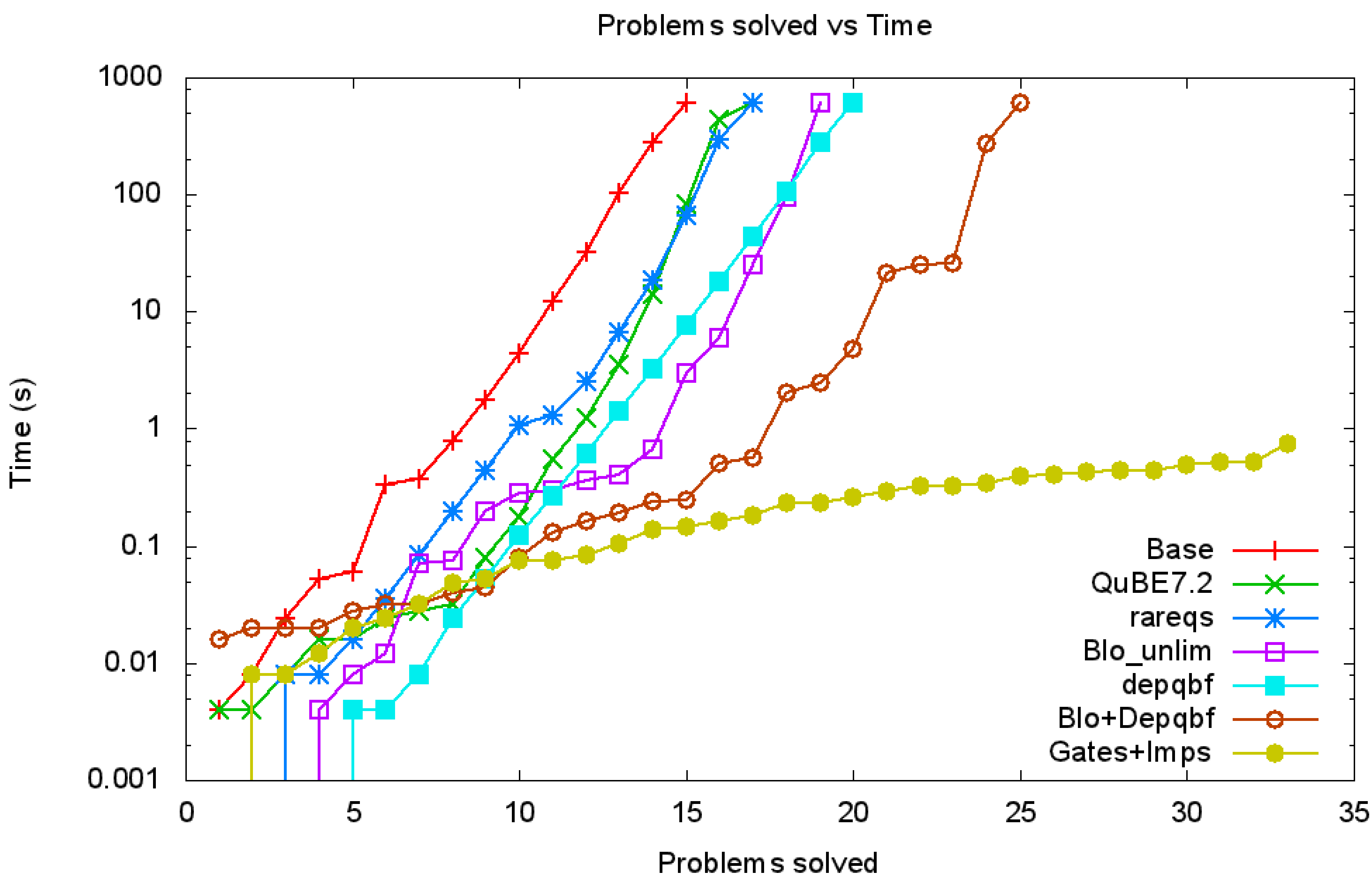
Extreme example

- A family of benchmarks with parameter n

$$
\begin{aligned}
&\exists e \forall x_1 x_2 x_3 \ldots x_n \\
&\qquad\qquad (x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus \ldots \oplus x_n) \\
&\qquad\quad \vee (e \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus \ldots \oplus x_n)
\end{aligned}
$$



Problems solved vs Time
Time (s)
Problems solved
Base
QuBE7.2
rareqs
Blo_unlim
depqbf
Blo+Depqbf
Gates+Imps
0
100
200
300
400
500
600
0
5
10
15
20
25
30
35



Problems solved vs Time
1000
100
10
1
0.1
0.01
0.001
Time (s)
0
5
10
15
20
25
30
35
Problems solved
Base
QuBE7.2
rareqs
Blo_unlim
depqbf
Blo+Depqbf
Gates+Imps

Conclusions

- CNF does not provide enough information to reason about solutions
- It is possible to use existing incomplete methods to partially reconstruct CNF. That information can be used such that:
  - The better the reconstruction, the more beneficial it is
  - If reconstruction is poor, efficiency is not lost
- Plaisted-Greenbaum encoding can also be reconstructed

Questions?