# Quantified Maximum Satisfiability: A Core-Guided Approach

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Sixteenth International Conference on
Theory and Applications of Satisfiability Testing

Helsinki, Finland
July 11, 2013

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Applications — optimization problems with quantified constraints.

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## Example

$$\xi = \exists e_1, e_2\ \forall x_1, x_2.\ (\neg e_1 \wedge \neg e_2) \rightarrow (x_1 \vee x_2)$$

# Quantified MaxSAT

$$\psi = \exists E\ \overrightarrow{Q}.\ \varphi$$

## Definition

Assignment $\mathcal{A}_E$ is a ***solution*** of $\psi$ iff $\overrightarrow{Q}.\ \varphi\big|_{\mathcal{A}_E}$ is true.

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Consider a *cost function* $f(e_1, \dots, e_l) = \sum_{i=1}^{l} a_i \cdot e_i$, $|E| = l$.
**Find** $\mathcal{A}_E \in \mathcal{M}(\psi)$ s. t. $\forall \mathcal{B}_E \in \mathcal{M}(\psi)$: $f(\mathcal{A}_E) \leqslant f(\mathcal{B}_E)$.

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$f(e_1, e_2) = 2 \cdot e_1 + 3 \cdot e_2$ $\quad$ $f(1,0) = \min_{\mathcal{M}(\xi)} f(e_1, e_2) = 2$

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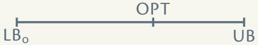

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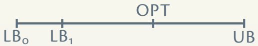

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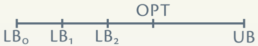

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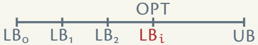

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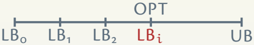



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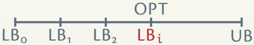


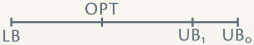


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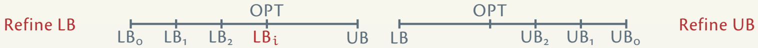

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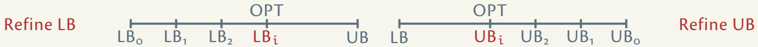

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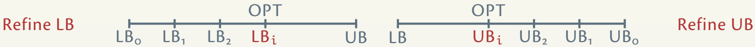


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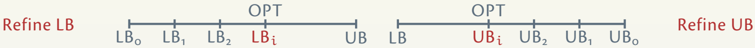


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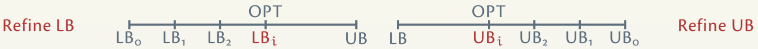


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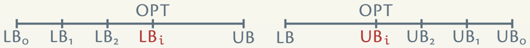


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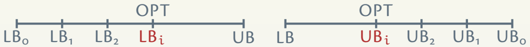


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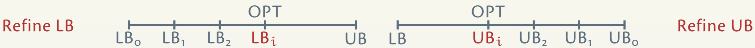


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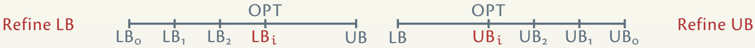


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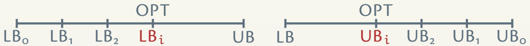


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**Definition**

Formula $\varphi_C = \varphi \wedge \varphi'_S$, $\varphi'_S \subseteq \varphi_S$, is an ***unsatisfiable core*** of $\psi'$ if $\exists E \, \overrightarrow{Q}. \; \varphi_C$ is false.

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**input** : $\psi = \exists E\ \overrightarrow{Q}.\ \varphi$, and $\varphi_S$

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R_all ← ∅                                        # set of all relaxation variables
while true:
    ψ'_R = ∃E ∃R_all →Q. φ ∧ φ_S
    (st, φ_C, A_E) ← QBF(ψ'_R)                    # calling a QBF oracle

    if st = true:
        return A_E

    R ← ∅
    foreach c ∈ Soft(φ_C):                        # relaxing the core
        let r be a new relaxation variable
        R ← R ∪ {r}
        φ_S ← φ_S \ {c} ∪ {c ∨ r}

    φ ← φ ∧ CNF(∑_{r∈R} r ≤ 1)                    # updating the hard part
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*Partial* expansion: $\exists X.\ \bigwedge_{\nu \in W} (\varphi_H \wedge \varphi_S)\big|_{\nu} \quad W \subseteq \{0,1\}^{|Y|}$

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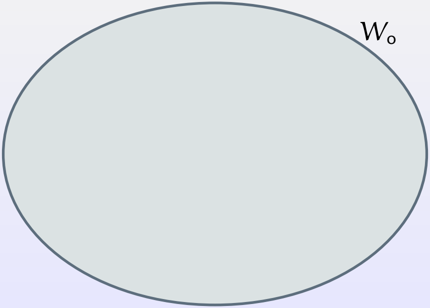

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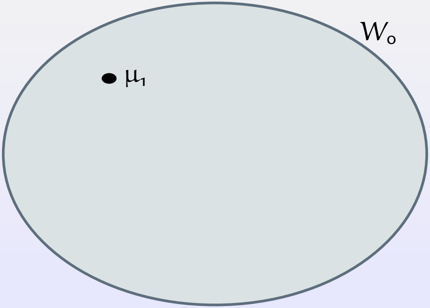

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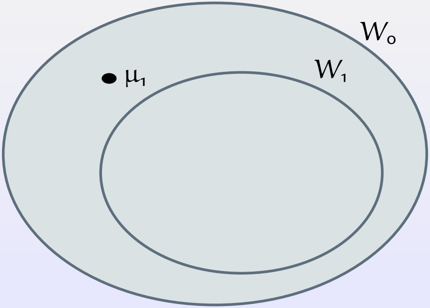

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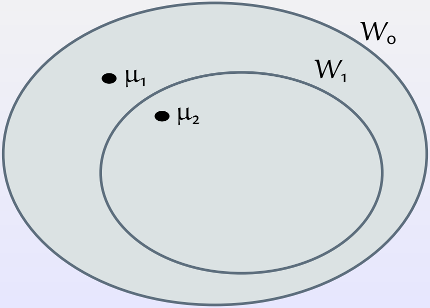

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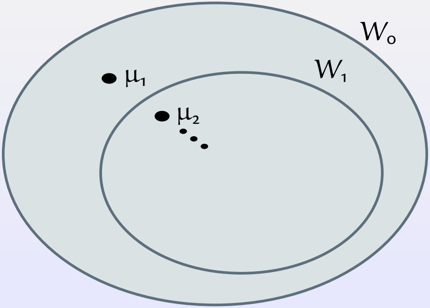

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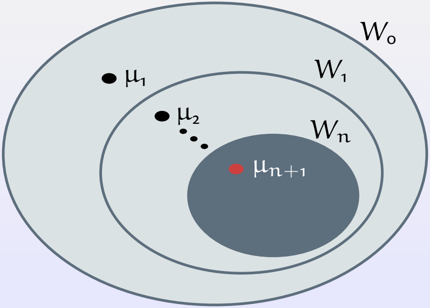

# Computing Cores in CEGAR-based 2QBF

**input** : $\exists X \forall Y.\ \varphi_H \wedge \varphi_S$

$\omega \leftarrow \emptyset$

**while** true:

  $\varphi \leftarrow \mathtt{CNF}(\bigwedge_{\nu \in \omega} \varphi_H\big|_\nu) \cup \bigwedge_{\nu \in \omega} \varphi_S\big|_\nu$

  $(st_1, \mu, \varphi_C) \leftarrow \mathtt{SAT}(\varphi)$ `# candidate`

  **if** $st_1 =$ false:

    $\varphi'_S \leftarrow \{c \in \varphi_S \mid c' \in \varphi_C, \nu \in \omega, c' = c\big|_\nu\}$

    **return** (false, $\varphi_H \wedge \varphi'_S$) `# no candidate found`

  $(st_2, \nu) \leftarrow \mathtt{SAT}\left(\neg(\varphi_H \wedge \varphi_S)\big|_\mu\right)$ `# counterexample`

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```
ω ← ∅
while true:
    φ ← CNF(⋀_{ν∈ω} φ_H|_ν) ∪ ⋀_{ν∈ω} φ_S|_ν
    (st₁, μ, φ_C) ← SAT(φ)                               # candidate
    if st₁ = false:
        φ'_S ← {c ∈ φ_S | c' ∈ φ_C, ν ∈ ω, c' = c|_ν}
        return (false, φ_H ∧ φ'_S)                        # no candidate found
    (st₂, ν) ← SAT(¬(φ_H ∧ φ_S)|_μ)                       # counterexample
    if st₂ = false:
        return (true, μ)                                  # solution found
    ω ← ω ∪ {ν}
```

# Smallest MUS Problem

### Definition

Formula $\psi^*, \psi^* \subseteq \varphi$, is called a ***smallest MUS*** of $\varphi$ if

1. $\psi^*$ is unsatisfiable
2. for any MUS $\psi, \psi \subseteq \varphi$, the following holds $|\psi^*| \leqslant |\psi|$

### Example

$\varphi = \{\ x_2 \vee \neg x_3 \vee \neg x_4,\ \ x_1 \vee x_2,\ \ x_3,\ \ \neg x_1,\ \ x_4,\ \ \neg x_2\ \}$

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  - $\exists S\, \forall X.\ \neg \varphi_R \wedge \varphi_S$, # core-guided search
    where $\varphi_S = \{\neg s_1, \ldots, \neg s_m\}$ # soft constraints

# Improvements of the Approach

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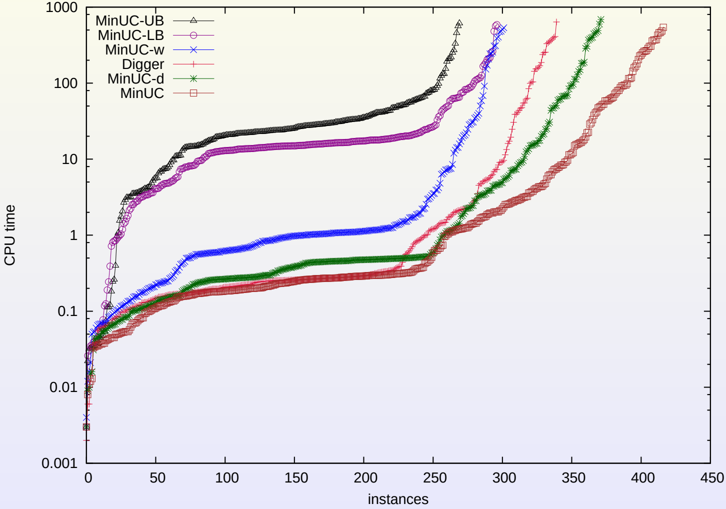
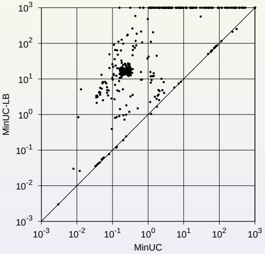
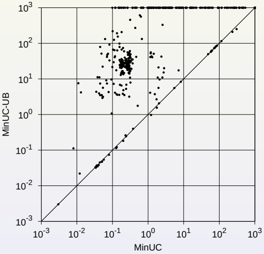
# Performance Comparison: MinUC vs Digger



# Performance Comparison: Linear Search vs Core-Guided

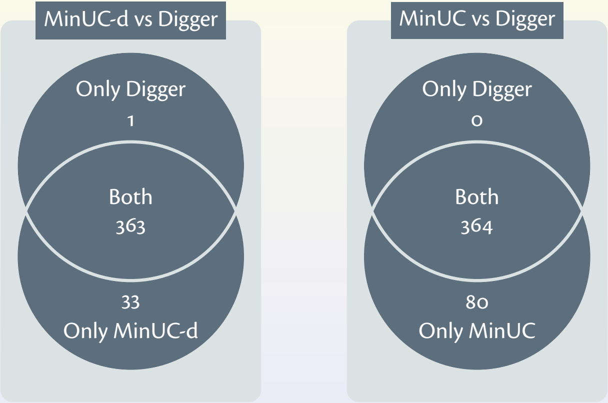



(a) MinUC vs MinUC-LB




(b) MinUC vs MinUC-UB

# Number of Solved Instances




Total number of instances — 682.

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# Thank you for your attention!