Exponential Separations in a Hierarchy of Clause Learning Proof Systems

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Resolution rule

If C, D are clauses with $x \in C$ and $\bar{x} \in D$, then

$$Res_x(C, D) := (C \setminus x) \vee (D \setminus \bar{x})$$

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A Resolution refutation of F is a derivation of the empty clause $\Box$ from F.

DPLL and Tree Resolution

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Wanted: Similar correspondence for DPLL with clause learning.

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  $C_v \in F$ or $C_v = C_u$ for some $u \prec v$ (lemma)

$\prec$ is the post-order on trees.

Clause learning and *RTL*

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A refutation R in RTL is in $RTL(k)$, if every lemma C used in R is of width $w(C) \leq k$.

Previous lower bounds

Theorem (BHJ 08)

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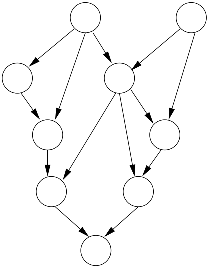
- $F_n^{(k)}$ *have* $RTL(k+1)$*-refutations of size* $n^{O(1)}$.
- $F_n^{(k)}$ *requires* $RTL(k)$*-refutations of size* $2^{\Omega(n/\log n)}$.

Graph Pebbling

Pointed DAG: G with in-degree 2 and one sink t.



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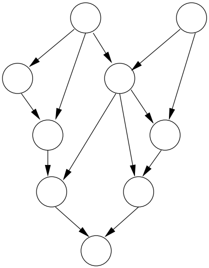

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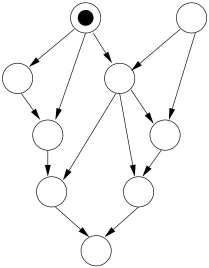

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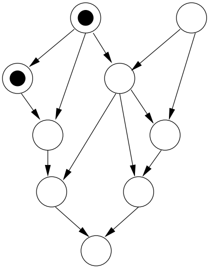

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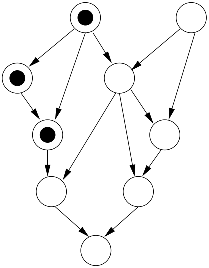

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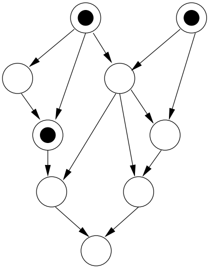

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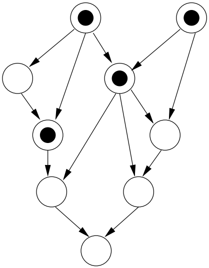

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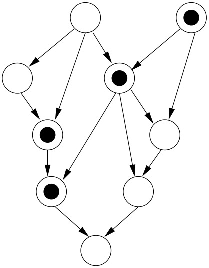

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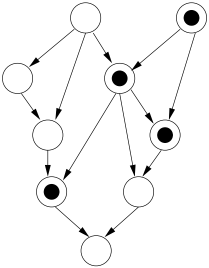

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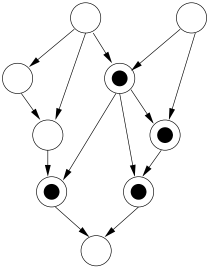

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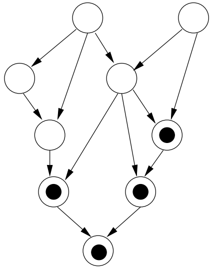

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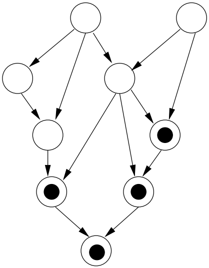

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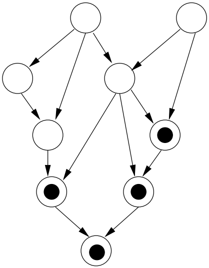

Pebbling number $\text{Peb}(G)$: min. # of pebbles in game on G.

Theorem (Celoni, Paul, Tarjan 1977)

There are dags G_n of size n with $\text{Peb}(G_n) \geq \Omega(n/\log n)$.

Pebbling formulas

Clauses $Imp(G)$:

x_v	for every source v
$x_u \wedge x_{u'} \to x_v$	for $(u, v), (u', v) \in E$
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Theorem (Ben-Sasson et al. 2004)

Every tree resolution refutation of $\mathit{Imp}^2(G)$ is of size $2^{\Omega(\mathrm{Peb}(G))}$.

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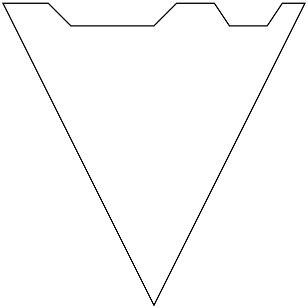
Theorem

For every G of size n,
$Imp^{\oplus k}(G)$ has $RTL(k)$-refutations of size $O(2^{3k}n)$.

Proof of the lower bound

- Let R be a refutation of $Imp^{\oplus(k+1)}(G)$



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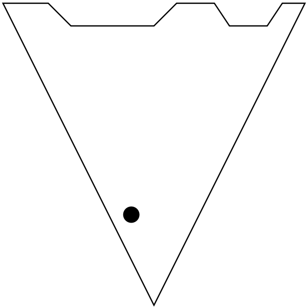

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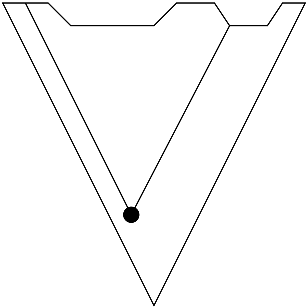

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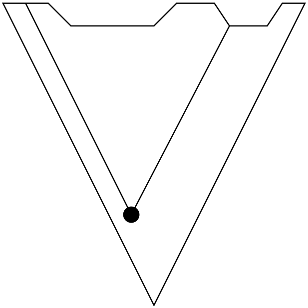

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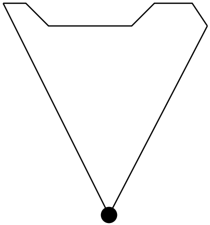

Generalized XORification

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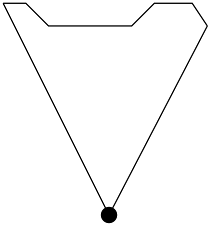
- Pick ρ with $C\lceil\rho = 0$
- $R_C\lceil\rho$ is refutation of $Imp^{\oplus(k+1)}(G)\lceil\rho$
- $Imp^{\oplus(k+1)}(G)\lceil\rho = Imp_{\beta}^{\oplus 2}(G)$



Lower bound

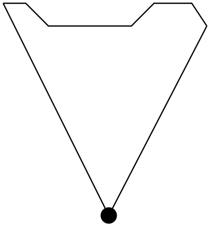
Generalization of the lower bound by Ben-Sasson et al.:

Theorem

Every tree resolution refutation of $Imp_{\beta}^{\oplus 2}(G)$ *is of size* $2^{\Omega(\mathrm{Peb}(G) - a(\beta))}$, *where* $a(\beta) := \#\{\, v \,;\, \beta_0(x_v) = 1 \,\}$

Proof of the lower bound

- Let R be a refutation of $Imp^{\oplus(k+1)}(G)$
- Find first C with $w(C) \leq k$
- Subtree R_C is tree-like derivation of C
- Pick ρ with $C \lceil \rho = 0$
- $R_C \lceil \rho$ is refutation of $Imp^{\oplus(k+1)}(G) \lceil \rho$
- $Imp^{\oplus(k+1)}(G) \lceil \rho = Imp_{\beta}^{\oplus 2}(G)$ for a β with $a(\beta) \leq 1$
- lower bound shows $|R| \geq |R_C| \geq 2^{\Omega(\mathrm{Peb}(G))}$



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Theorem

For every k, the formulas $Imp^{\oplus(k+1)}(G_n)$

- *have $RTL(k+1)$-refutations of size $n^{O(1)}$.*
- *require $RTL(k)$-refutations of size $2^{\Omega(n/\log n)}$.*