Experiments with Reduction Finding

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The Magic of SAT

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Beyond SAT

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$$\textbf{find } r : \forall x \, (x \in P \leftrightarrow r(x) \in Q)$$

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Questions

- how do we represent r,P,Q, and x?
- how do we approach the problem? **(CEGAR vs QBF vs ASP)**
- how do current tools perform?

How do we represent reductions?

Representing Reductions

reduction $r : \; \forall x \, (x \in P \leftrightarrow r(x) \in Q)$

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Standard reductions

- r is a (ptime, logspace, …) **Turing machine**
- x is a **word**
- P,Q are sets of words given by **Turing machines**

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Reductions in logic

- r is a (quantifier-free, first-order, …) **query**
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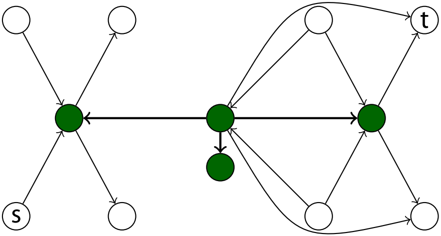
Question: is there a useful correspondence?

Relational Structures and Logics

Relational Structures $\mathfrak{A} = (A, \mathbf{R}_1^{\mathfrak{A}}, \mathbf{R}_2^{\mathfrak{A}}, \ldots, \mathbf{R}_l^{\mathfrak{A}}, \mathbf{c}_1^{\mathfrak{A}}, \ldots, \mathbf{c}_m^{\mathfrak{A}})$



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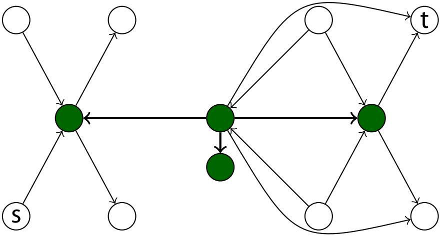

First-Order and Second-Order Logic over $\sigma = \{\mathbf{E}\}$

The graph is a clique (FO): $\forall x, y(x = y \vee \mathbf{E}(x, y))$

The graph is 3-colourable (∃SO):

$$\exists R, G, B\big(\forall x, y\,(R(x) \vee G(x) \vee B(x)) \wedge \big(\mathbf{E}(x, y) \rightarrow \neg\,((R(x) \wedge R(y)) \vee (G(x) \wedge G(y)) \vee (B(x) \wedge B(y)))\big)$$

Descriptive Complexity

FO Interpretations (Queries) $\theta = (k, \varphi_0, \psi_1, \ldots, \psi_m)$

- k is the dimension
- $\varphi_0(x_1, \ldots, x_k)$ defines the new universe
- $\psi_i(x_1, \ldots, x_{kr_i})$ define the new relations

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Example: $\big(k = 2,\ \varphi_0 = \top,\ \psi_1(x_1, x_2, y_1, y_2) = \mathbf{E}(x_1, x_2) \wedge (y_1 = y_2 \vee y_2 = \mathbf{s})\big)$

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Complexity classes under interpretations (Immerman)

- **quantifier-free reductions** are weaker than ptime
- **still** P=NP iff SAT $\leq_{qf}$CVP
- **and** NL=NP iff SAT $\leq_{qf}$REACH,
- **and** coNL=NL **(true)** iff $\neg$REACH $\leq_{qf}$REACH

How do we find reductions?

Existential SO using SAT Solvers

Transformation $\exists\text{SO} \ni \varphi, \mathfrak{A} \quad \rightsquigarrow \quad \psi$ **Boolean**

$$\mathbf{Rel}(a_1, \ldots, a_k) \rightsquigarrow \top /_{\mathfrak{A}} \bot \qquad Var(a_1, \ldots, a_k) \rightsquigarrow X_{Var, a_1, \ldots, a_k}$$

$$\varphi_1 \wedge \varphi_2 \rightsquigarrow \hat{\varphi}_1 \wedge \hat{\varphi}_2 \qquad \exists x \varphi \rightsquigarrow \bigvee_{a \in \mathfrak{A}} \hat{\varphi}(a) \qquad \forall x \varphi \rightsquigarrow \bigwedge_{a \in \mathfrak{A}} \hat{\varphi}(a)$$

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Model-Checking

(1) transform $\varphi, \mathfrak{A} \rightsquigarrow \psi$

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Example: 3-colouring a graph

$$\exists R, G, B \left(\forall x, y \left(R(x) \vee G(x) \vee B(x) \right) \wedge \left(\mathbf{E}(x, y) \rightarrow \neg \left((R(x) \wedge R(y)) \vee (G(x) \wedge G(y)) \vee (B(x) \wedge B(y)) \right) \right) \right)$$

Two Basic Applications

Example (and counter-example) finding

$$\varphi \in \mathrm{FO},\ n \in \mathbb{N} \quad \rightsquigarrow \quad \mathfrak{A} \mid |\mathfrak{A}| = n \text{ and } \mathfrak{A} \models \varphi \quad (\text{or } \mathfrak{A} \models \neg\varphi)$$

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Formula finding

$$\text{outline of } \varphi,\ \mathfrak{A} \quad \rightsquigarrow \quad \varphi \mid \mathfrak{A} \models \varphi$$

Outline: formula with Boolean atom guards. Example:

$$X_1\mathbf{E}(x_1, x_1) \wedge X_2\mathbf{E}(x_1, x_2) \wedge X_3\mathbf{E}(x_2, x_1) \wedge X_4\mathbf{E}(x_2, x_2) \wedge$$
$$X_5\neg\mathbf{E}(x_1, x_1) \wedge X_6\neg\mathbf{E}(x_1, x_2) \wedge X_7\neg\mathbf{E}(x_2, x_1) \wedge X_8\neg\mathbf{E}(x_2, x_2)$$

Automatic Reduction Finding

Assumptions: outline of θ and the maximal $|\mathfrak{A}|$ fixed

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Finding reductions by CEGAR

- Find a l-DNF reduction θ_i good on counter-examples $\mathfrak{E}_0, \ldots, \mathfrak{E}_i$
- Find a counter-example $\mathfrak{E}_{i+1}$ to θ_i, iterate

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$$\exists\theta\ \forall\mathfrak{A}\ (\mathfrak{A} \models \varphi_P \ \leftrightarrow\ \theta(\mathfrak{A}) \models \varphi_Q)$$

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Easy example: s-t reachability to strongly connected (both NL-complete)

$$\mathrm{Reach} = [\mathrm{tc}_{x,y}\, \mathbf{E}(x,y)](.s,.t) \qquad \mathrm{SC} := \forall x, y(\mathrm{tc}_{x,y}\, \mathbf{E}(x,y))$$

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$$(k = 1,\; \varphi_0 = \top,\; \psi_1 = x_1 = \mathbf{s} \vee x_2 = \mathbf{t} \vee \mathbf{E}(x_2, x_1))$$

How do current tools perform?

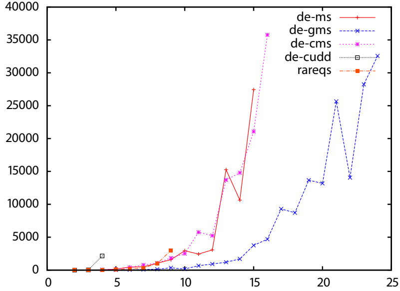
Reduction Finding Results

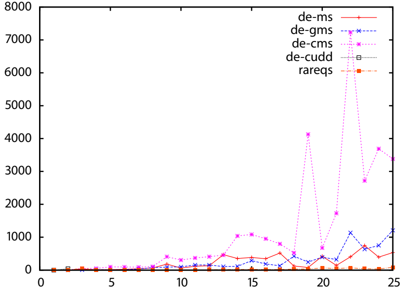
 # Unsolved cases out of $48 \times 48 = 2304$: **CEGAR** vs **QBF** vs **ASP** (claspD)

(c,n)	$(1,3)$	$(2,3)$	$(3,3)$	$(1,4)$	$(2,4)$	$(3,4)$
de-gms	0	0	10	0	5	103
de-cudd	0	116	537	0	186	722
rareqs	0	0	16	19	65	204
depqbf	0	142	547	16	297	711
qube	10	536	949	82	760	1082
cirqit	58	673	1138	511	1092	1357
cirqit’	157	523	903	–	–	–
skizzo	522	1058	1156	975	1327	1434
gringo	40	393	590	72	593	836
lparse	51	396	605	75	635	850
RedFind	1	152	396	2	347	547

CEGAR Results

Performance on ¬REACH to REACH, $k = 1$, scaling n (left) and c (right)





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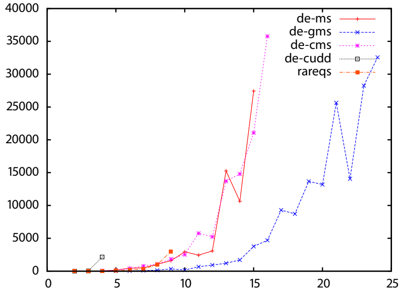

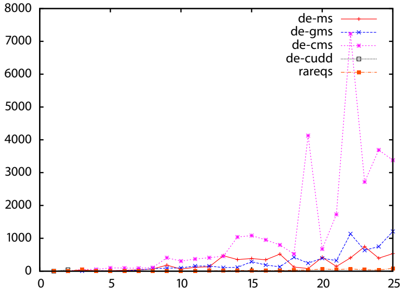

Increasing dimension to $k = 2$

	de-ms	de-gms	de-cms	de-cudd	rareqs
$k = 1, c = 1, n = 3$	0.05	0.06	0.08	0.07	0.03
$k = 2, c = 1, n = 2$	0.06	0.11	0.28	6.30	0.06
$k = 2, c = 1, n = 3$	3562.14	1696.26	1755.03	timeout	3267.10

Outlook

Beyond SAT: **find** $f : \forall x$ **"**$f(x)$ **is good"**

What can we do?

- simple evaluation and reduction finding
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Other possible applications

- Finding LFP formulas for **NP $\cap$ coNP** properties
- Early results on unary 1-variable 1-LFP
  - reachability games in < 1 minute **(answer: yes)**
  - parity games in < 1 hour **(answer: no)** (cf. Dawar, Grädel, CSL'08)

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