# Efficient Clause Learning for Quantified Boolean Formulas via QBF Pseudo Unit Propagation

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**Conflict-Driven Clause Learning (CDCL):** [SS96]

- Crucial for the performance of modern SAT solvers.
- Resolution proofs, trimming the search space.
- Extensions of CDCL for SAT to QBF: QCDCL.

**Traditional QCDCL for QBF:** [ZM02, GNT02, GNT06, Let02]

- Like CDCL is based on resolution, QCDCL is based on Q-resolution.
- Q-resolution derivation of the clause to be learned.
- Tautological resolvents must be avoided explicitly.

**Problem:**

- Common approach to avoiding tautologies in traditional QCDCL has an exponential worst case [VG12].
- The derivation of a single learned clause might have an exponential number of intermediate resolvents.

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# Overview (2/2)

**Our Work:** efficient polynomial time procedure for QCDCL.

- QCDCL based on *QBF Pseudo Unit Propagation (QPUP)* [VG12]: carefully select the order of resolution steps in QCDCL to avoid tautologies.
- Learn a single non-tautological clause in polynomial time.
- QPUP-based QCDCL is compatible with other approaches (e.g. Alexandra's talk).
- Implementation in the search-based QBF solver DepQBF.

**Syntax**

- Prenex CNF: quantifier-free CNF over quantified Boolean variables.
- PCNF $\psi := Q_1 x_1 \ldots Q_n x_n.\ \phi$, where $Q_i \in \{\exists, \forall\}$, no free variables.
- $Q_i x_i \leq Q_{i+1} x_{i+1}$: variables are linearly ordered.

**Example**

A CNF: $(x \vee \neg y) \wedge (\neg x \vee y)$, and a PCNF: $\forall x \exists y.\ (x \vee \neg y) \wedge (\neg x \vee y)$.

**Search-based QBF Solving with Clause Learning:**

- Implicitly enumerate paths in a semantic tree by recursive variable instantiation.
- Terminology "QCDCL": conflict-driven clause learning (CDCL) for QBF.
- Learn clauses at unsatisfiable (i.e. conflicting) branches in the search tree.
- Like CDCL in SAT: QCDCL is based on resolution for QBF.

# Resolution for QBF

**Q-Resolution:**

- Combination of universal reduction and propositional resolution.
- Sound and refutational-complete proof system for QBF: Q-resolution proofs.

**Definition ([BKF95])**

Given a clause $C$, *universal reduction (UR)* on $C$ produces the clause

$$UR(C) := C \setminus \{l \in L_\forall(C) \mid \forall l' \in L_\exists(C) : var(l') < var(l)\},$$

where $<$ is the linear variable ordering given by the quantifier prefix.

- Universal reduction deletes trailing universal literals from clauses.

**Definition ([BKF95])**

- Let $C_1$, $C_2$ be non-tautological clauses where $v \in C_1, \neg v \in C_2$ for an $\exists$-variable $v$.
- *Tentative Q-resolvent* of $C_1$ and $C_2$: $C_1 \otimes C_2 := (UR(C_1) \cup UR(C_2)) \setminus \{v, \neg v\}$.
- If $\{x, \neg x\} \subseteq C_1 \otimes C_2$ for some variable $x$, then no Q-resolvent exists.
- Otherwise, the non-tautological *Q-resolvent* is $C := UR(C_1 \otimes C_2)$.

# Boolean Constraint Propagation for QBF (QBCP)

- Generate assignments by assumptions, unit clause rule, **universal reduction (UR).**
- Like BCP for SAT: antecedent clauses and implication graphs.
- Like CDCL for SAT: QCDCL is based on the implication graph given by QBCP.

## Example (assignments, implication graphs)

```
p cnf 5 4
e 1 3 4 0
a 5 0
e 2 0
-1 2 0
3 5 -2 0
4 -5 -2 0
-3 -4 0
```

Implication graph:

Assignment $A := \{\}$.

- Assumption: $A := A \cup \{1\}$.
- Clause (-1 2) is unit under $A$
  $A := A \cup \{2\} = \{1, 2\}$
  $ante(2) :=$ (-1 2)
- Clause (3 5 -2) is unit under A and UR.
  $A := A \cup \{3\} = \{1, 2, 3\}$
  $ante(3) :=$ (3 5 -2)
- Clause (4 5 -2) is unit under A and UR.
  $A := A \cup \{4\} = \{1, 2, 3, 4\}$
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- Clause (-3 -4) is conflicting under $A$.
  $ante(\emptyset) :=$ (-3 -4)

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$1 \longrightarrow 2$

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$$1 \longrightarrow 2 \longrightarrow 3$$

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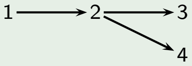


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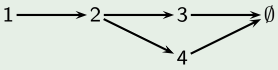


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- Start at conflicting clause, resolve on existential variables *in reverse assignment order* until the resolvent is asserting (i.e. will be unit after backtracking).
- Resolve on existential variables which were assigned as unit literals, using clauses (i.e. antecedents) which became unit during QBCP.
- Tautological resolvents might occur but must be avoided by "resolving around": $\Rightarrow$ deviate from strict reverse assignment order [GNT06].
- Worst case exponential number (in $|IG|$) of intermediate resolvents [VG12].

## Example (continued)

```
p cnf 5 4
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e 2 0
-1 2 0
3 5 -2 0
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```
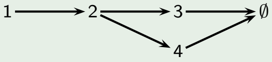
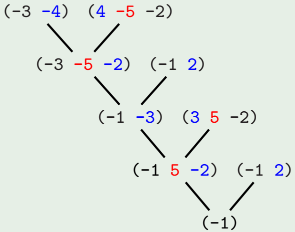

Clause (-3 -4) conflicting:




Assignment $A = \{1, 2, 3, 4\}$
Assignment order: 1, 2, 3, 4
Resolve on: 4, 2, 3, 2.

Derivation of learned clause (-1):

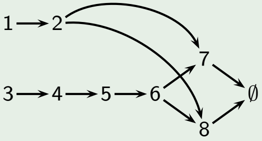


**QBF Pseudo Unit Propagation (QPUP):** [VG12]

- Basic idea: given an implication graph (IG), associate the conflict node $\emptyset$ and each variable $x$ assigned by the unit literal rule with a "QPUP clause" $qpup(x)$.
- Walking through the entire IG in assignment ordering, compute $qpup(x)$ by resolving $ante(x)$ with already computed $qpup(y)$ s.t. $\neg y \in ante(x)$.
- Resolve in assignment ordering: tautologies cannot occur by construction.
    - Compare: traditional QCDCL resolves in reverse assignment ordering.
- Finally, the non-tautological and asserting QPUP clause $qpup(\emptyset)$ related to the conflict node $\emptyset$ can be learned.

**Example (to be continued)**

Assumptions: 1, 3
Assignment order: 1, 2, . . . , 8.

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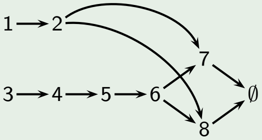

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The clause $qpup(\emptyset)$ = (-1 -3) is non-tautological and asserting and can be learned.

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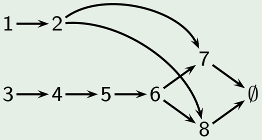


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*qpup*(∅) = (-1 -3)

*qpup*(2) := *ante*(2) = (-1 2)

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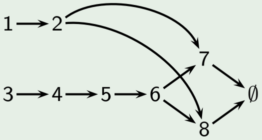


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# Towards Efficient QCDCL (2/3)

## Example (continued; computing QPUP clauses)

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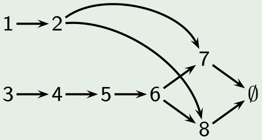


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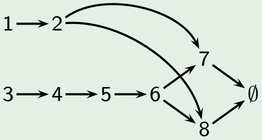


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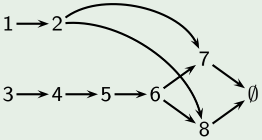


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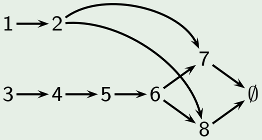


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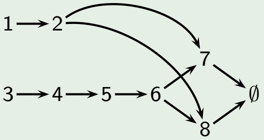


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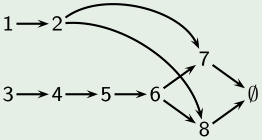


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# Towards Efficient QCDCL (3/3)

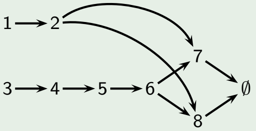
**Problem:**

- Computing QPUP clauses for every $n \in IG$: total $|IG|$ resolution steps.
- Traversal starts at assumption nodes $\Rightarrow$ full traversal, prohibitive at each conflict.
- Goal: find alternative start points closer to the conflict node $\emptyset$.

**Unique Implication Points (UIPs):**

- Nodes in the implication graph which are on every path from the most recent assumption to the conflict node $\emptyset$.
- Comprehensive theory in SAT CDCL [SLM09].
- A UIP is a good candidate as a start point to compute QPUP clauses.

**Example (continued)**




- Node 6 is the first UIP (i.e. closest to $\emptyset$).
- Node 5 is the second UIP.
- Node 4 is the third UIP.
- Node 3 is the fourth UIP.

# Efficient QPUP-based QCDCL (1/2)

**Two-Phase Algorithm:**

- Phase 1: starting at the conflict node $\emptyset$, walk back through the implication graph in reverse assignment order to find suitable start points.
  - Focus on finding UIPs.
  - In general, a single UIP as a start point is not enough.
  - At the latest, phase 1 terminates when reaching the assumption nodes.
- Phase 2: compute the QPUP clauses $qpup(x)$ for all nodes $x$ reachable when walking from the start points found in phase 1 towards the conflict node $\emptyset$.
  - Unlike in traditional QCDCL, here resolutions are done in assignment order.

**Goal:**

- The non-tautological and asserting QPUP clause $qpup(\emptyset)$ of the conflict node $\emptyset$ computed in phase two will be learned.
- Challenge: what nodes are suitable start points?

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- The non-tautological and asserting QPUP clause $qpup(\emptyset)$ of the conflict node $\emptyset$ computed in phase two will be learned.
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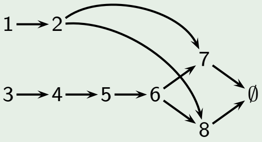
## Example (computing QPUP clauses from start points)

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p cnf 10 7
e 1 3 4 5 7 8 0
a 10 0
e 2 6 0
(-1 2),
(-3 4),(-4 5),(-5 6),
(7 10 -2 -6),(8 -10 -2 -6),
(-7 -8)
```

Node 6 is the 1-UIP, $\{7, 8, \emptyset\}$ reachable, *qpup*($\emptyset$) = (10 -10 -2 -6) tautological.

Node 5 is the 2-UIP, $\{6, 7, 8, \emptyset\}$ reachable, but *qpup*($\emptyset$) = (-5 10 -10 -2).

Node 4 is the 3-UIP, $\{5, 6, 7, 8, \emptyset\}$ reachable, but *qpup*($\emptyset$) = (-4 10 -10 -2).

Node 3 is the 4-UIP, $\{4, 5, 6, 7, 8, \emptyset\}$ reachable, but *qpup*($\emptyset$) = (-3 10 -10 -2).

$\Rightarrow$ **impossible to use a UIP as the single start point.**

Observe:

Node 5 is the 2-UIP, but *qpup*($\emptyset$) = (-5 10 -10 -2) is tautological.

$\Rightarrow$ must eventually resolve on variable 2 to avoid tautology.

**Nodes $\{1, 5\}$ are suitable start points:**

$\{2, 6, 7, 8, \emptyset\}$ reachable and *qpup*($\emptyset$) = (-1 -5)

Compare:

Using the assumptions $\{1, 3\}$ as trivial start points produces *qpup*($\emptyset$) = (-1 -3).

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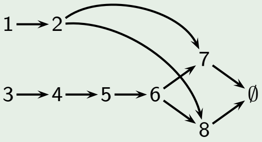


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⇒ **impossible to use a UIP as the single start point.**

Observe:
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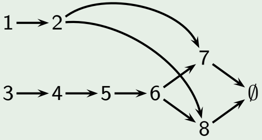
# Efficient QPUP-based QCDCL (2/2)

**Example (computing QPUP clauses from start points)**




```
p cnf 10 7
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Node 3 is the 4-UIP, $\{4, 5, 6, 7, 8, \emptyset\}$ reachable, but *qpup*($\emptyset$) = (-3 10 -10 -2).

$\Rightarrow$ **impossible to use a UIP as the single start point.**

Observe:

Node 5 is the 2-UIP, but *qpup*($\emptyset$) = (-5 10 -10 -2) is tautological.

$\Rightarrow$ must eventually resolve on variable 2 to avoid tautology.

**Nodes $\{1, 5\}$ are suitable start points:**

$\{2, 6, 7, 8, \emptyset\}$ reachable and *qpup*($\emptyset$) = (-1 -5)

Compare:

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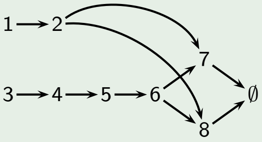
# Efficient QPUP-based QCDCL (2/2)

## Example (computing QPUP clauses from start points)

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Node 3 is the 4-UIP, $\{4, 5, 6, 7, 8, \emptyset\}$ reachable, but $qpup(\emptyset)$ = (-3 10 -10 -2).
⇒ **impossible to use a UIP as the single start point.**

Observe:
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⇒ must eventually resolve on variable 2 to avoid tautology.

**Nodes $\{1, 5\}$ are suitable start points:**
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Compare:
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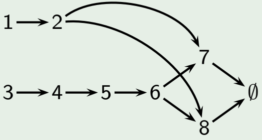
## Example (computing QPUP clauses from start points)

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$\Rightarrow$ **impossible to use a UIP as the single start point.**

Observe:

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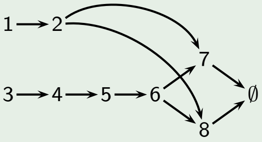
# Efficient QPUP-based QCDCL (2/2)

## Example (computing QPUP clauses from start points)

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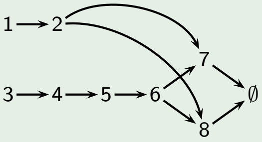
## Example (computing QPUP clauses from start points)



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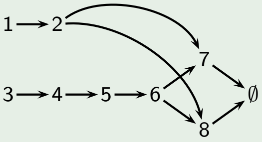


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**Implementation:**

- Search-based, clause-learning QBF solver DepQBF.
- Features: traditional QCDCL and QPUP-based QCDCL.
- Our implementation is more sophisticated than the procedure sketched before.
- No QPUP clauses are computed during the search for start points.

## Example (formula class with exponential traditional QCDCL [VG12])

Each formula in this class can be decided by learning a single unit clause. The derivation of that learned clause by traditional QCDCL has an exponential number of resolution steps.

| Size Parameter | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Traditional QCDCL | 6 | 14 | 30 | 62 | 126 | 254 | 510 | 1022 | 2046 | 4094 |
| QPUP-based QCDCL | 6 | 10 | 14 | 18 | 22 | 26 | 30 | 34 | 38 | 42 |

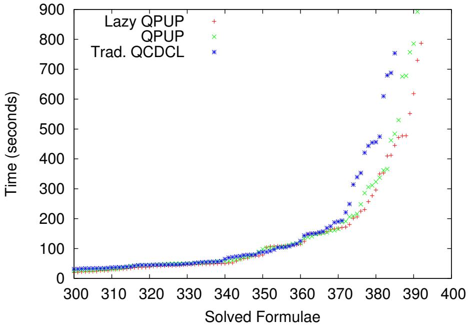
Table: number of resolutions in DepQBF to derive the single learned unit clause.

**Benchmarks from Previous QBF Evaluations:**

- Improvements with QPUP-based QCDCL.
- Lazy QPUP-based QCDCL: learn a clause without explicitly deriving it.
  - Conservatively predict the set literals definitely in the learned clause.
- Further experimental results: see the QBF Gallery 2013.

| QBFEVAL'10 (568 formulas, no preprocessing) | |
|---|---|
| Lazy QPUP | 393 (170 s, 223 u) |
| QPUP | 392 (170 s, 222 u) |
| Trad. QCDCL | 386 (167 s, 219 u) |

- Instances solved (sat, unsat).
- Intel Xeon E5450, 3.00 GHz, timeout 900 seconds, 8 GB memory limit.



# Conclusions

**Traditional QCDCL for QBF:**

- Based on implication graphs resulting from QBCP.
- Start at conflict node, resolve on variables in reverse assignment order.
- Tautologies must be avoided explicitly: exponential worst case.

**QPUP-based QCDCL:**

- Start at internal nodes of the implication graph, resolve on variables in assignment order working towards the conflict node.
- With the right set of start point, tautologies cannot occur by construction.
- For practical efficiency: finding start points close to the conflict node.
- Compatible with other approaches in search-based QBF solving.

**Future Work:**

- Procedural improvements.
- More detailed comparison of QCDCL variants (traditional, QPUP, lazy QPUP).

New version of DepQBF to be released: `http://lonsing.github.com/depqbf/`

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